

“Two dimensional QCD is a string theory”

Review on references [1] and [4]

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Abstract

This text, written as dissertation within the M.Sc. course in particle theory at the Centre for Particle Theory, University of Durham, during the academic year 1993/94, reviews two articles by D.Gross and by D.Gross and W.Taylor which interpret the $1/N$ - expansion of the partition function of QCD_2 as string perturbation series. For this required mathematical and physical background is presented.

Foreword

The task set for this dissertation was to give a review on the recent development of an interpretation of pure Quantum Chromodynamics in two dimensions (QCD_2) as a string theory.

The work in this direction was started by D.J. Gross [1] , inspired, presumably, by an article of G. 't Hooft [2] , and is still in progress.

The mathematical and physical concepts used for performing the underlying idea that indeed QCD_2 can be described as a string theory stem from group theory, the theory of Riemann surfaces, loops in algebraic topology and lattice gauge theories, and, of course, from Yang-Mills and string theories.

A complete treatment of the progress done up to this time proved to be impossible because of the extensive literature published very recently and using tools I could not be familiar with at the start of my work.

Rather than to give an overview of all achievements I decided to introduce as thoroughly as possible the concepts, tools and ideas used in the main references. I will, therefore, follow the development through the article of D.Gross [1], the first article by D.Gross and W.Taylor [4], and the article by J.Minahan [3]. As motivation for the relevance of planar diagrams and two dimensional pure gauge theory for the description of the strong interaction and its main characteristic, the confinement, I will give G.'t Hooft's $1/N$ -expansion [2] and his derivation of a "meson" spectrum [5] .

Additionally the dissertation contains short accounts of the necessary results from group theory, algebraic topology, Riemann surfaces in particular, and lattice gauge theory.

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Chapter 1

1.1 Introduction

The purpose of this chapter is to give reasons for seeking another perturbation theory/ quantisation procedure for the strong interaction (section 1.1), to connect the resulting $1/N$ -approach to planar diagrams (QCD_2 , section 1.2), and to give an example why the physics contained in this concept might be nontrivial and physically true: 't Hooft's derivation of a “meson” spectrum (section 1.3).

By now the perturbation theory of QCD as a weak coupling expansion seems to describe correctly those contributions to high-energy scattering, which are due to the strong interaction.

Although this QCD is very successful in the high-energy- (UV-) limit, it fails to describe the gauge theory in the infrared- (IR-) limit: The renormalised coupling stops being small at “large” distances (ca. 1fm). This is qualitatively in good agreement with the experimental “observation” that no single quarks or coloured states appear in the detectors, but it does not provide a quantitative description of QCD in the IR, e.g. to calculate masses and excitations of hadrons and thereby to check QCD at the other end of momentum scale.

This is in spite of the fact that the quark model was invented to analyse the huge spectrum of boundstates observed as resonances in high-energy experiments and that non abelian gauge theories were proposed to describe confinement.

The spectrum of mesons is divided into families with linear correlation of squared mass and spin (Regge-trajectories, see e.g. [6]). Of course we would like to have a theory to calculate this spectrum, even more because it appears to be simpler than the atomic spectra which lead to the development of Quantum Mechanics. More generally the situation is very much the same as in the beginning of this century: Without an analytic model for the spectrum the description of nature is unsatisfactorily incomplete.

Lattice gauge theory has already provided contributions to the understanding of QCD in the IR (see eg. [7] [8] [9]). There was made the observation that in the strong coupling expansion (cf. section 2.1) the free energy can be expressed as a sum over surfaces with quite complicated contact terms, though (see references in [1] under “[2]”). D. Gross takes this as an “existence proof” of a string formulation of QCD [1] . But lattice gauge theories can not provide an analytic description of QCD.

In contrast to the gauge coupling, which gives no good expansion due to its growth at low momentum, there is hope, that there exists a perturbation theory with $(\pi N)^{-1}$ as effective expansion parameter [10]. This corresponds to regarding $N = 3$ as “large”; the “limit” $N \rightarrow \infty$ which is frequently used to determine the right $1/N$ expansion of the partition function has to be understood in this sense.

In the recent decades “string theories” have been proposed as theories of the strong interaction (“dual models”) and as “theories of everything”. Although a lot is known about those theories it seems that there is not much, by now, whose description is actually given by a string theory [11] .

Planar diagrams as in ’t Hooft’s paper [2] and strings share the property of being Riemann surfaces (or triangulations of those, respectively) [12] .

Both the representation theory of the gauge groups $SU(N)$ and $U(N)$ and the classification of Riemann surfaces can be connected to the representation theory

of the symmetric group of permutations S_n (see chapters 3 and 5). The partition function of QCD_2 is solved exactly as “Fourier-”series over representations (see chapter 2, [13], [14]).

To summarise there is a necessity to find a theory for the IR-limit , there is hope that a planar expansion with $1/N$ as perturbative parameter might be the concept to be applied, there are tools to connect QCD_2 with a “string” perturbation theory, and there is the hope that by the knowledge about string theories we can find a string field theory giving this expansion and which leads to some theory in higher dimension.

In this context I will try to present the work of D.Gross and D.Gross & W.Taylor published in [1][4] alongside mathematical and physical background.

Still within this chapter the particular limit for the $1/N$ - expansion chosen by D.Gross is motivated following ’t Hooft’s work and argumentation is given that two dimensional gauge models have physically relevant features like confinement and a linear spectrum: ’t Hooft’s 2D meson model is used to argue that considering two dimensions could teach lessons about higher dimensions.

Chapter 2 develops the exact solution of QCD_2 using the heat kernel action.

Chapter 3 gives the necessary concepts and results from group theory used in the later chapters.

Some discussion on one of the central terms in this approach, the “Wilson loop average”, is given in chapter 4.

Chapter 5 presents the tools from the theory of Riemann surfaces required later on.

The results achieved then are used to give the first step of the interpretation: The understanding of the $1/N$ - expansion of the QCD_2 - partition function as a “chiral” sum over covering maps from strings onto the QCD_2 - spacetime manifold. (chapter 6)

The $1/N$ - expansion is improved for the limit $N \rightarrow \infty$ by introducing two coupled “chiral” sectors in chapter 7. The “non chiral sum” is interpreted then.

Chapter 8, finally, outlines the treatment of bordered surfaces in this framework.

1.2 't Hooft's $U(N)$ model

The purpose of this section is to present a gauge model showing domination of planar diagrams in the limit $N \rightarrow \infty$ and indicating which specific $N \rightarrow \infty$ limit one has to take to find the “right” $1/N$ expansion for eg. QCD_2 . In the next section a two dimensional version of this model will be considered to derive an approximately linear spectrum of some “ μ^2 ”.

Because the usual generators of the Lie algebra $\mathfrak{u}(N)$ of $U(N)$ do not have to be traceless as those of $\mathfrak{su}(N)$ the Feynman rules give rise to a simpler group theory (“colour”) factor for “index” (coloured) loops. In fact it is simply N , which simplifies the treatment of the model considerably.

The parameter N is treated to be free ($N \rightarrow \infty$) in order to determine a reasonable $1/N$ expansion under the assumption that $N = 3$ is a large number.

In the following first step the quantisation of the $U(N)$ model is outlined and Feynman rules are given to the required extend. The second next step is to relate higher order diagrams to their dominantly planar character in the $N \rightarrow \infty$ limit, and to give the specific limit used by D.Gross [1].

1.2.1 Feynman rules

The gauge model chosen is $U(N)$ gauge theory. The similarities and differences between this model and QCD ($SU(N)$ model) can be extracted from a comparison of the Lie algebras $\mathfrak{u}(N)$ and $\mathfrak{su}(N)$.

$u(N)$ and $su(N)$ both consist of anti hermitean matrices and satisfy the same algebra: [20][21]

$$[A_i^j, A_k^l] = \delta_k^j A_i^l - \delta_i^l A_k^j \quad (1.1)$$

where $su(N)$ consists of traceless matrices only i.e. $u(N) \cong u(1) \oplus su(N)$ and we have one abelian gauge field more for $u(N)$.

Lie algebra valued fields, i.e. gauge fields and ghosts, carry in both models two colour “currents”; these are the same as the two colour lines of gluons in QCD. With the standard choice of a basis for $u(N)$ [21] [20] each index stands for one out of N “currents”.

The perturbation theory and the Feynman rules follow, formally, after the temporary introduction of external sources associated with each field (see [18]) from the equation

$$Z(\{J_\omega\}) = e^{-S_I(-\frac{\delta}{\delta J_A}, -\frac{\delta}{\delta J_{\bar{\psi}}}, -\frac{\delta}{\delta J_\psi}, -\frac{\delta}{\delta J_{\bar{\eta}}}, -\frac{\delta}{\delta J_\eta})} Z_0(J_A, J_{\bar{\psi}}, J_\psi, J_{\bar{\eta}}, J_\eta) \quad (1.2)$$

This equation holds in euclidean spacetime. Z_0 is the partition function given by the path integral weighted by the quadratic terms of the effective Lagrangean only ([16] sect.10.5). This path integral can be solved in analogy with the following Gaussian integrals in finite dimensions:

$$\begin{aligned}
& \int d\bar{x}_1 dx_1 d\bar{x}_2 \dots dx_n \exp(-x^+ \mathbf{A} x + \eta^+ x + x^+ \eta) \\
& = -\det(\mathbf{A}) \exp(\eta^+ \mathbf{A}^{-1} \eta)
\end{aligned} \tag{1.3}$$

for anticommuting variables

$$\begin{aligned}
& \int d\bar{z}_1 dz_1 d\bar{z}_2 \dots dz_n \exp(-z^+ \mathbf{B} z + \rho^+ z + z^+ \rho) \\
& = (2\pi)^n |\det(\mathbf{B})|^{-1} \exp(\rho^+ \mathbf{B}^{-1} \rho)
\end{aligned} \tag{1.4}$$

for commuting variables

S_I contains the remaining terms which in turn are drawn outside the the path integral like parameter differentiations of the partition function in statistical mechanics.

The external sources share the functional features of the fields they are associated with, but they are external. For the moment this spoils the gauge invariance, but the objects to be calculated will be gauge invariant after the limit $J_\omega \rightarrow 0$, so the formalism will be consistent after this limit [17].

After the choice of a unique and complete set of indices for each family of functions, eg a basis for the Lie algebra, the conservation laws at the vertices follow formally from [19]

$$\frac{\delta J_\omega(x')}{\delta J_\Omega(x)} = \delta_{\omega,\Omega} \delta(x' - x) \tag{1.5}$$

So with the usual choice for $U(N)$ [21],[20] the propagators of the $U(N)$ model carry for the gauge field and the ghosts two currents of one out of N colours each. These currents pass vertices independently. By this any current/index loop produces a colour factor $\sum_i \delta_i^i = N$ for the amplitude of the diagram. This is simpler than for QCD and the reason for using $U(N)$ rather than $SU(N)$.

The possibility of assigning two independent currents to the gauge and ghost fields makes planar diagrams as considered in section 1.2.2 orientable.

The additional abelian gauge field associated with $U(1)$ in $U(N)$ does not affect the nonabelian character of the algebra, so the model should give the right implications in the confining IR-limit.

't Hooft formulates the model in euclidean spacetime, i.e. after a “Wick-”rotation, which does not affect the validity of the theory nor changes the rules (or their derivation) significantly. 't Hooft chooses the Feynman gauge, such that the rules look like the usual ones from QCD (see [16] [15]).

We do not need the Feynman rules in detail (see [2]). Their geometric character and the power of the coupling constant κ attached to the vertices will be sufficient for the discussion in section 1.2.2.

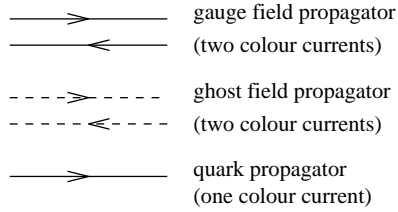


fig. 1.1: the propagators

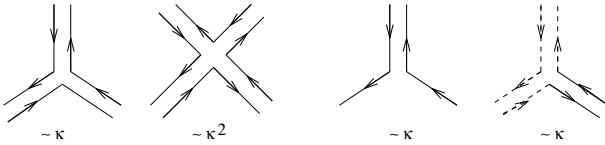


fig. 1.2: the vertices

1.2.2 Planar diagrams & large N limit

Here we will consider connected diagrams of higher order in the gauge coupling κ corresponding to the IR-limit, where κ is no longer small enough to describe the theory correctly with diagrams involving a few vertices only.

We want to treat the diagrams as geometrical entities under the assumption that the powers of N i.e. the number of index loops associated with a diagram by the Feynman rules make an implication on their importance in the IR-limit.

Because we are still dealing with the expansion in powers of κ we have to recognise the associated power of κ , too.

For this the external legs of the diagram are removed by gauge invariant source functions, eg. (f, f' are flavour indices)

$$J = \sum_i \bar{\psi}^{fi} \psi_i^{f'}$$

By this there is no colour entering or leaving the diagram; the colour flow continues at J . The source looks like the creation/annihilation of a meson (see figure 1.3).

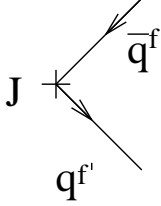


fig. 1.3: source function

Now we take the diagram and interpret each propagator as an edge, gauge and ghost field propagators are internal edges: Every index-loop is a face. The resulting polyhedron is finished by taking each quark-loop as a face, too.

For the planar diagrams this gives a surface, for which the following calculation holds:

Be F, E, V, L, I the numbers of faces, edges, vertices, quark loops, and index loops in the diagram. Be V_n the number of n -point vertices.

Then are, by construction:

$$F = L + I \quad V = \sum_i V_i \quad 2E = \sum_n nV_n$$

Each index loop contributes a factor N , each 3-point vertex a factor κ , each 4-point vertex a factor κ^2 .

This amounts to an overall factor associated to the diagram reading as:

$$r = \kappa^{V_3+2V_4} N^I = \kappa^{2E-2V} N^{F-L}$$

Because the surfaces are orientable the Euler-Poincaré-characteristic χ is given by

$$\chi = V - E + F = 2 - 2g \quad [22]:\text{Theo.2.49} \quad (1.6)$$

where g is the number of holes (“genus”) in the surface of which the diagram is the triangulation.

This leads to

$$r = (\kappa^2 N)^{\frac{1}{2}V_3+V_4} N^{2-2g-2L} \quad (1.7)$$

Under the fundamental assumption that a reasonable $1/N$ -expansion for a gauge model like QCD exists this formula implies the following:

- Quark loops are comparatively suppressed for $N \rightarrow \infty$
- Planar diagrams arrange themselves according to their genus
- The limit $N \rightarrow \infty$ and $\kappa^2 N = \text{const.}$ might lead to a good $1/N$ -expansion.

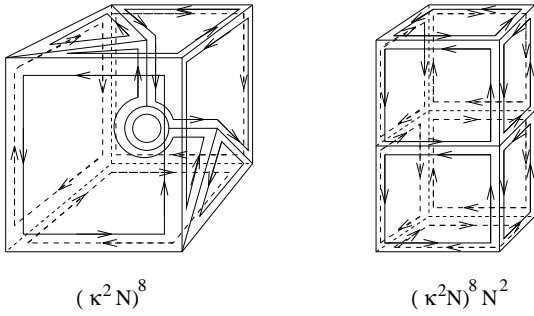


fig. 1.4: comparison planar and nonplanar diagram

Regarding the power of N and κ associated with a diagram as a weight for its contribution leads to the conclusion, that planar diagrams are dominating when $N \rightarrow \infty$. It should be sufficient to examine a simple, nontrivial example of two diagrams with the same set of vertices in $d=3$, one is planar, one is not (see figure 1.4). This shows that nonplanar diagrams are suppressed because the flow through the “interior” of the diagram admits less loops.

The indication that considering planar gauge theories could be useful will be supported furtherly in the next section.

1.3 Two dimensional $U(N)$ meson model

In the following treatment of an “amusingly simple model” (’t Hooft [5]) two characteristics of strong interaction appear: Single quarks have infinite mass (confinement) and there is a discrete, linear spectrum connected with quark pairs (Regge- trajectories see eg.[6])

The procedure followed in this section will be such:

- in 1.3.1 the very convenient Feynman rules are developed
- section 1.3.2 solves the only dressed propagator
- and finally, in 1.3.3, we see approximate solutions of an hermitean eigenvalue equation, which one could regard as something like the Klein-Gordon equation.

1.3.1 Feynman rules

The model is a $U(N)$ gauge theory in two euclidean dimensions.

The Lagrangean leading to the Feynman rules is the usual:

$$L = \frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^i - \bar{\psi}^{fi} (\gamma_\mu D_\mu + m_{(f)}) \psi_i^f$$

$$F_{\mu\nu}^j = \partial_\mu A_{\nu}^j - \partial_\nu A_{\mu}^j + \kappa[A_\mu, A_\nu]^j_i$$

$$D_\mu \psi_i^f = \partial_\mu \psi_i^f + \kappa A_{i\mu}^j \psi_j^f$$

$$A_{i\mu}^j(x) = -A_{j\mu}^{*i}(x)$$

The indices i, j are Lie algebra indices, μ, ν are spacetime indices, f is the flavour index, ψ quark fields, A the antihermitean gauge field.

There are two euclidean coordinates on the plane: x_0, x_1 . Lightcone coordinates $x_\pm = \frac{1}{\sqrt{2}}(x_1 \pm x_0)$ prove to be convenient.

A notion of upper and lower indices and a summation convention are introduced as follows:

$$x_1 \equiv x^1 \quad x_0 \equiv -x^0$$

$$x^\mu p_\mu \equiv x^\mu p^\mu \equiv x_\mu p_\mu \equiv x_+ p^+ + x_- p^- \equiv x^+ p^- + x^- p^+ \equiv x_+ p_- + x_- p_+$$

The algebra for the γ -matrices may be determined using the lightcone versions of the Dirac- and the Klein-Gordon- equation (as usual) and is:

$$\gamma_-^2 = \gamma_+^2 = 0 \quad \{\gamma_+, \gamma_-\} = 2 \quad (1.8)$$

The gauge condition is an axial gauge, the lightcone gauge: $A_- = A^+ = 0$

Which leads to $F_{+-} = -\partial_- A_+$ and

$$L = -\frac{1}{2} \text{tr}(\partial_- A_+)^2 - \bar{\psi}^f (\gamma \partial + m_{(f)} + \kappa \gamma_- A_+) \psi^f$$

There is no coupling to ghosts in this gauge (cf. [15]).

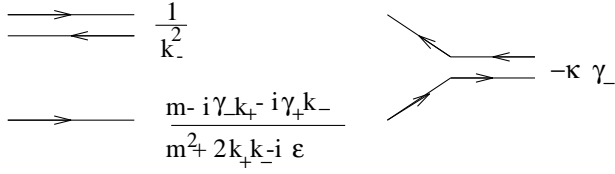


fig. 1.5: Feynman rules for meson-model

Because of the simple γ -algebra and the fact that there is only a single vertex in the theory the γ -matrices disappear from the theory whenever vertices are connected to propagators. A simple calculation leads to an equivalent set of simpler rules (see figure 1.6).

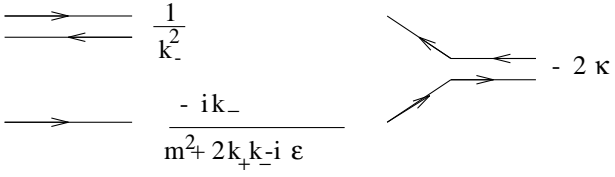


fig. 1.6: effective Feynman rules

In the following we will ignore contributions from quark loops in agreement with the observation in the previous section.

1.3.2 THE dressed propagator

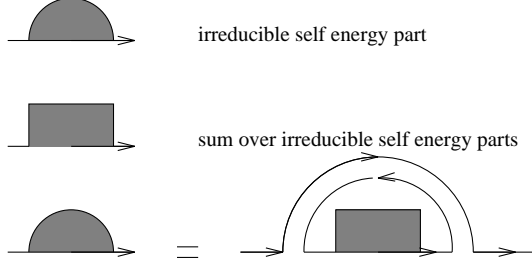


fig. 1.7: equation for dressed propagator

As usual we need the dressed propagators of the theory, i.e. two point Green's functions including all self energy parts of the particle (cf.[16], [15]). The gauge field propagator is already “dressed” because we have no coupling between gauge fields and we neglect quark loops. So *the* only one to be dressed is the quark propagator.

From the form of the undressed propagator we already see the form of the dressed propagator:

$$\frac{-ik_-}{m^2 + 2k_+k_- - k_- \Gamma(k) - i\epsilon}$$

where $i\Gamma(k)$ stands for the sum of one-point-irreducible self energy parts.

The model is very simple and we can find $i\Gamma(k)$ by solving a quite simple equation it *has* to satisfy (see figure 1.7).

$$i\Gamma(p) = \frac{4\kappa^2}{(2\pi)^2 i} \int dk_+ dk_- \frac{1}{k_-^2} \frac{-i(k_- + p_-)}{m^2 + [2(k_+ + p_+) - \Gamma(k + p)](k_- + p_-) - i\epsilon}$$

We observe that k_+, p_+ only appear in the constellation $k_+ + p_+$. The integral will be invariant under the shift $k_+ + p_+ \rightarrow k_+$ after, possibly, a regulation business, such that $\Gamma(p)$ is independent of p_+ .

The UV-divergence in the k_+ integral, as it can be seen easily by choosing a symmetric cutoff, factorises. This factor is not involved with the following calculations and can be ignored henceforth (cf.section 2.2).

The k_+ -integral is, up to the divergence, given by:

$$\frac{\pi i}{2|k_- + p_-|}$$

This gives for the k_- -integration an IR-divergence. The choice of a symmetric cutoff $\pm\lambda$ leads to a cancellation of all λ from all formulae at a later stage of the calculation; so presumably its choice does not affect the final result.

Performing the k_- -integral in this manner leads to:

$$\Gamma(p) = \Gamma(p_-) = -\frac{\kappa^2}{\pi} \left(\frac{\sigma(p_-)}{\lambda} - \frac{1}{p_-} \right) \quad (1.9)$$

$\sigma(p_-)$ is the sign of p_-

And the dressed propagator is:

$$\frac{-ik_-}{m^2 - \frac{\kappa^2}{\pi} + 2k_+k_- + \frac{\kappa^2|k_-|}{\pi\lambda} - i\epsilon} \quad (1.10)$$

In the limit $\lambda \rightarrow 0$ the pole of the propagator is shifted towards $k_+ \rightarrow \infty$. This means that there are no free quarks in this model: They would have infinite mass.

As mentioned above the cutoff λ will not be involved in the “meson” spectrum calculated below. This means that quarks are confined to the neighbourhood of (an-)other quark(-s).

1.3.3 Derivation of the approximate spectrum

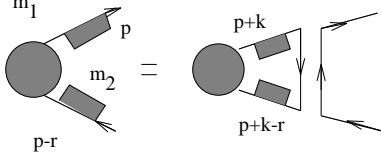


fig. 1.8: equation for meson spectrum

For “blobs” out of which a quark with mass m_1 and momentum p and an antiquark with mass m_2 and momentum $r - p$ appear exists, due to the simple Feynman rules of this model, an equation, depicted in figure 1.8, which will lead to the spectrum. ψ stands for such an arbitrary vertex about which we do not need to know much more than that it has to satisfy the equation below.

$$\begin{aligned}
 \psi(p, r) = & -\frac{4\kappa^2}{(2\pi)^2 i} \times \\
 & (p_- - r_-)[M_2^2 + 2(p_+ - r_+)(p_- - r_-) + \frac{\kappa^2}{\pi\lambda}|p_- - r_-| - i\epsilon]^{-1} \times \\
 & p_-[M_1^2 + 2p_+p_- + \frac{\kappa^2}{\pi\lambda}|p_-| - i\epsilon]^{-1} \times \\
 & \iint dk_+ dk_- \frac{\psi(p+k, r)}{k_-^2} \tag{1.11} \\
 M_i^2 = & m_i^2 - \frac{\kappa^2}{\pi}
 \end{aligned}$$

Writing $\phi(p_-, r) = \int dp_+ \psi(p_+, p_-, r)$ leads to:

$$\begin{aligned}
\phi(p_-, r) &= -\frac{\kappa^2}{(2\pi)^2 i} \times \\
&\int dp_+ [p_+ - r_+ + \frac{M_2^2}{2(p_- - r_-)} + (\frac{\kappa^2}{2\pi\lambda} - i\epsilon) \sigma(p_- - r_-)]^{-1} \times \\
&[p_+ + \frac{M_1^2}{2p_-} + (\frac{\kappa^2}{2\pi\lambda} - i\epsilon) \sigma(p_-)]^{-1} \times \\
&\int dk_- \frac{\phi(p_- + k_-, r)}{k_-^2}
\end{aligned} \tag{1.12}$$

The integral can be evaluated by summing over the poles in the upper half plane according to:

$$\int_{-\infty}^{+\infty} dx R(x) = 2\pi i \sum_{\text{Im}(z)>0} \text{res}_z R \tag{1.13}$$

In case *both* poles lie in the upper half plane, the sum is always zero: A simple calculation shows, that the two residua cancel each other. So the nontrivial case is when the poles lie on different sides of the real axis, i.e. $\sigma(p_-) = -\sigma(p_- - r_-)$. We may take $r_- > 0$.

Then:

$$\begin{aligned}
\phi(p_-, r) &= \frac{\kappa^2}{2\pi} \Theta(p_-) \Theta(r_- - p_-) \times \\
&[\frac{M_1^2}{2p_-} + \frac{M_2^2}{2(p_- - r_-)} + \frac{\kappa^2}{\pi\lambda} + r_+]^{-1} \times \\
&\int dk_- \frac{\phi(p_- + k_-, r)}{k_-^2}
\end{aligned} \tag{1.14}$$

where the usual theta-functions $\Theta(x)$ keep track of the position of the poles.

Using the principal value \wp and the Sokhotsky-Plemelj-formula [25] [24]:

$$\wp \int dk_- \frac{\phi(k_-)}{k_-^2} = \frac{1}{2} \int dk_- \frac{\phi(k_- + i\epsilon)}{(k_- + i\epsilon)^2} + \frac{1}{2} \int dk_- \frac{\phi(k_- - i\epsilon)}{(k_- - i\epsilon)^2} \quad (1.15)$$

$$\int dk_- \frac{\phi(p_- + k_-, r)}{k_-^2} = \frac{2}{\lambda} \phi(p_-) + \wp \int dk_- \frac{\phi(p_- + k_-, r)}{k_-^2} \quad (1.16)$$

Now we use all this and dimensionless variables:

$$\alpha_{1,2} = \frac{\pi M_{1,2}^2}{\kappa^2} = \frac{\pi m_{1,2}^2}{\kappa^2} - 1$$

$$-2r_+ r_- = \frac{\kappa^2}{\pi} \mu^2 \quad \frac{p_-}{r_-} = x$$

which leads to

$$\mu^2 \phi(x) = \left(\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x} \right) \phi(x) - \wp \int_0^1 dy \frac{\phi(y)}{(y-x)^2} \equiv \mathbf{H} \phi \quad (1.17)$$

Two observations have to be made:

- The dependence on the cutoff λ does no longer *appear* to exist.
- The solutions of this equation should satisfy the equation (1.14) it was derived from. We have therefore to examine the solutions which vanish at 0 and 1 and have support only within $[0,1]$.

Interpreting this equation as an eigenvalue equation of the operator \mathbf{H} defined by this equation one can easily show that for two solutions ψ, ϕ we have:

$$(\psi, \mathbf{H} \phi) = (\phi, \mathbf{H} \psi)$$

(by application of the definition, the usual scalarproduct and the observation about the support of ψ, ϕ)

Now we solve the equation approximately:

$$\wp \int_0^1 dy \frac{e^{i\omega y}}{(y-x)^2} \approx \wp \int_{-\infty}^{+\infty} dy \frac{e^{i\omega y}}{(y-x)^2} \quad (x \in [0, 1])$$

$$\approx \pi i e^{i\omega x} \omega$$

by using the definition of \wp and the formula:

$$\int dx R(x) e^{ix} = 2\pi i \sum_{\text{Im} z > 0} \text{res}_z (R(\zeta) e^{i\zeta}) \quad [23] \text{p153, S.6.2}$$

For $\alpha_{1,2} \approx 0$ the following functions approximately solve the equation:

$$\phi^k(x) \simeq \sin k\pi x \quad k \in \mathbb{N} - \{0\}$$

$$\Rightarrow \mu_k^2 \approx \pi^2 k$$

So within the limits of the approximations made we have a linear, discrete spectrum of an hermitean operator **H**.

This model may serve as a motivation that we might carry nontrivial physics through a treatment based on two dimensional Yang-Mills-theory.

Chapter 2

The Heatkernel Action

This chapter is devoted to the derivation of the exact solution of Yang-Mills theory in two dimensions, especially for the case of QCD_2 .

The solution stems from concepts introduced for the lattice approximation of the theory. Therefore some background from lattice gauge theory will be given first.

2.1 Wilson's action

In the usual continuum Yang-Mills theory a gauge field is introduced to define a connection which makes the derivative covariant under gauge transformations. The gauge field then tells how to transport vectors along paths in spacetime.

Equally well one could start with a set of parallel transporters which act on the vector along the path and determine the result of parallel transportation from an initial state of the vector.

From the gauge field point of view these parallel transporters are given by “pathordered exponentials” ([22]p.337, [26]). The Lie algebra contains the generators of infinitesimal gauge transformations. A parallel transporter can therefore be regarded as a successive application of Lie algebra operators along a path C

in the limit of zero steplength. The order of those successive operations has to be defined for nonabelian algebras. This defines the “pathordered exponential”, which actually has characteristics of an exponential map:

$$\mathbf{P}exp(\int_C \mathbf{A}_\mu dx^\mu)$$

On a lattice the path integral quantisation involves integration over all gauge transformations at all links between each two neighbouring points on the lattice. It does not make much sense to use gauge fields as the variables of this procedure. One rather uses the variables which are the actual variables of integration. For this an appropriate action has to be found to weight the configurations in terms of these variables.

I wish give an example, Wilson’s action, and its continuum limit to illustrate that actions like this one define an approximation which might be equivalent to the more usual continuum theory in the limit of small lattice spacing a . Lattice gauge theories therefore provide models for cases where the weak coupling perturbation theory does not work.

Wilson’s action is, like the heat kernel action, formulated in terms of group transformations associated with each elementary closed path on the lattice, i.e. around the “plaquettes” which are for a cubic lattice simply rectangles with edges of length a .

It is [27] [7]

$$S_j = \beta_j \sum_p \frac{1}{N} \frac{1}{2} tr(\mathbf{U}_p + \mathbf{U}_p^\dagger) = \beta_j \sum_p \frac{1}{N} Re(tr(\mathbf{U}_p))$$

where N is the gauge group parameter, the sum is over all plaquettes in the lattice and \mathbf{U}_p is the paralleltransporter once along the edges of the plaquette, sometimes called the “holonomy” (cf. section on Riemann surfaces).

β_j is a free parameter which is chosen to give the right continuum limit; regarding the action as a weight in the partition function of the lattice (like in solid state physics) β_j is called the “temperature parameter”.

The continuum limit is found by the observation, that the traces are simultaneously maximal for the identity [27]. Cutting the lattice into a maximal tree and the condition $\mathbf{U}_p = \mathbf{1} \forall p$ leads to a configuration where all holonomies associated with an edge are given by pure gauge.

\mathbf{U}_{ij} be such an edge transformation, then there exist two gauge transformations $g_i(g_j)$ at the points $i(j)$ on the lattice such that:

$$\mathbf{U}_{ij} = g_i g_j^{-1} \text{ [27]}$$

The further procedure is to perturb this configuration by introduction of a gauge field, i.e.

$$\mathbf{U}_{ij} = g_i \mathbf{P} \exp\left(\int_i^j \mathbf{A}_\mu dx^\mu\right) g_j^{-1}$$

and to use the Baker-Campbell-Hausdorff formula [32] [33] to derive \mathbf{U}_p . then the lattice spacing is send to zero, differences are replaced by derivatives, summations by integrations leading to (in two dimensions)

$$S_j \approx \frac{\beta_j \kappa^2 a^2}{N} \int dx^2 \left(-\frac{1}{4\kappa^2} \text{tr}(F_{\mu\nu}^2)\right)$$

The factor in front of the integral has to be unity to give the right continuum limit:

$$\beta_j = \frac{N}{\kappa^2 a^2}$$

(For this reason the strong coupling expansion is “natural” for lattice gauge theory [8].)

2.2 Wilson loop average for YM_2

Because the heat kernel action is determined in the continuum limit by another object, the Wilson loop average, we first have to define this object and to solve it in two dimensions.

The Wilson loop average is defined as the expectation value of

$$tr(\mathbf{P}exp(-\oint_C dx^\mu \mathbf{A}_\mu)) \quad (2.1)$$

This quantity is gauge invariant.

For the calculation below, following [28], we will use an axial gauge, which can always be found by usage of the path ordered exponential (see eg. [34]p.566f.).

The Wilson loop average in two euclidean dimensions and free gauge field is:

$$W[C] = \frac{1}{N} \int [D \mathbf{A}] tr(\mathbf{P}exp(-\oint_C dx^\mu \mathbf{A}_\mu)) exp \int \frac{1}{4\kappa^2} tr(F_{\mu\nu}^2) \quad (2.2)$$

Where N stands for the usual normalisation, $[D \mathbf{A}]$ for the path integral and $W[C]$ is the Wilson loop average of the closed path C.

Taking a gauge transformation

$$g(x_1, x_2) = \mathbf{P}exp(-\int_0^{x_2} dx^2 \mathbf{A}(x_1, x_2)_2)$$

which satisfies [22]

$$g^{-1} \frac{\partial}{\partial x_2} g = -g^{-1} \mathbf{A}^2 g$$

therefore we have

$$\mathbf{A}'^2 = g^{-1} \mathbf{A}^2 g + g^{-1} \frac{\partial}{\partial x_2} g = 0$$

The field strength is then given by $tr(\partial_2 \mathbf{A}_1)^2$.

The integral is solved in analogy with the integral

$$\begin{aligned} & \int d\bar{z}_1 dz_1 d\bar{z}_2 .. dz_n exp(-z^+ \mathbf{B} z + \rho^+ z + z^+ \rho) \\ &= (2\pi)^n |det(\mathbf{B})|^{-1} exp(\rho^+ \mathbf{B}^{-1} \rho) \end{aligned}$$

This leads to

$$W[C] = tr \mathbf{P}exp(-\frac{\kappa^2}{2} \oint_C \oint_C dx^1 dy^1 G(x, y) \mathbf{T}_{x^1}^a \mathbf{T}_{y^1}^a) \quad (2.3)$$

where G satisfies:

$$\partial_2^2 G(x, y) = \delta(x, y)$$

for which in commuting variables the solution is [19]

$$G(x, y) = \frac{1}{2} |x^2 - y^2| \delta(x^1 - y^1)$$

The Green's function is “singular”: it gives “infinite” contribution for any pair x, y with $y^1 - x^1 = 0$. In this sense the divergence of the k_+ -integral in section 1.3.2. can be understood.

$\mathbf{T}_{x^1}^a$ is a basis vector in the Lie algebra, which carries a location index due to the path ordering.

We now evaluate the exponential for a rectangle (see figure 2.1).

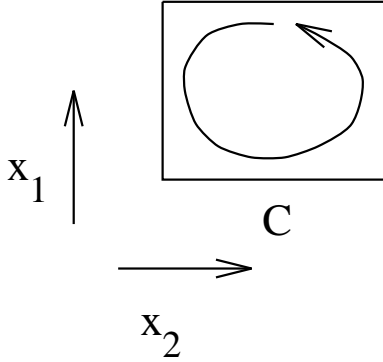


fig. 2.1: evaluation of loop integral

Along the x_2 edges we have no contribution and the path ordering leads to $\mathbf{T}_{x^1}^a \mathbf{T}_{x^1}^a = C_2$ all along the path, where C_2 is the quadratic Casimir (see next chapter).

The integral is (-1) times the area A of the enclosed rectangle.

This leads to:

$$W[C] = \text{tr}(\exp(\frac{\kappa^2}{2} C_2 A)) \quad (2.4)$$

So far there was no need to specify a representation R of the Lie algebra. If we choose a particular irreducible representation we have (see next chapter):

$$W[C] = \text{tr}_R \exp(\frac{\kappa^2}{2} C_2(R) A \mathbf{1}_R) = \exp(\frac{\kappa^2}{2} C_2(R) A \dim(R))$$

$\dim(R)$ is the dimension, $C_2(R)$ the Casimir-eigenvalue of the representation.

The next section will show that the choice of a rectangle implies no restriction, because the YM_2 -action is invariant under area-preserving diffeomorphisms; so the “area law” stated above will hold for all Wilson loops with no self intersections¹.

2.3 Additional symmetry of pure YM_2

In two dimensions there is another symmetry of the free YM-action. We follow the treatment given in [29].

$$S_{YM} = \frac{1}{4\kappa^2} \int_{\Sigma} \text{tr}(\mathbf{F} \wedge * \mathbf{F})$$

where Σ is the spacetime surface, \mathbf{F} is the field strength, $*\mathbf{F}$ stands for the Hodge star of \mathbf{F} , \wedge is the wedge product of differential forms (for definitions: [22]).

Choosing an area form μ on Σ one can write \mathbf{F} with a Lie algebra valued function ϕ as:

$$\mathbf{F} = \phi \mu$$

and following the definition of the Hodge star

$$\phi = *\mathbf{F}$$

This is in components with g being the determinant of the spacetime metric:

$$F_{ij}^a = \sqrt{|g|} \epsilon_{ij} \phi^a$$

¹Remark: G.West [35] found a relation between the Wilson loop average and the gluon propagator $D_{\mu\nu}^{ab}$ in the general case:

$$W[C] \leq \exp\left(\frac{\kappa^2}{2} \oint_C dx^\mu \oint_C dy^\nu D_{\mu\nu}^{ab} \delta_{ab}\right)$$

ϵ_{ij} is the totally antisymmetric tensor (Levi-Cevita symbol) in two dimensions, carries all of the tensorial character of \mathbf{F} and thereby admits the invariance:

$$S_{YM} = \frac{1}{4\kappa^2} \int_{\Sigma} \mu \text{tr}(\phi\phi) = S_{YM} = \frac{1}{4\kappa^2} \int_{\Sigma} dx^2 \sqrt{|\det g_{ij}|} \text{tr} \phi^2 \quad (2.5)$$

Because ϕ is a scalar any diffeomorphism $\Sigma \rightarrow \Sigma$ which leaves the area invariant will leave the action invariant.

2.4 Solution of QCD_2

2.4.1 Orthogonality of characters

Although some more group theory is yet to come we need some of it now.

The character of a group element is the trace of its representative in some linear representation of the group. Of particular interest are characters in irreducible unitarian representations (cf. next chapter).

Compact (or “unimodular”) groups have the nice feature to admit an invariant normalisable measure (the “Haar” measure) such that statements from the representation theory of finite groups can often be transferred to compact Lie groups using the correspondence

$$\frac{1}{|G|} \sum_{g \in G} \leftrightarrow \int dU \quad \int dU = 1$$

$U(N)$ and $SU(N)$ are unimodular (see [31]).

The proof of the orthogonality and completeness of irreducible characters given in [30] (Th.3.5, 3.7) generalises and leads to:

$$\int dU \chi_R(U) \chi_S(U^+) = \delta_{RS} \quad (2.6)$$

Formulae proved in the same way are:

$$\int dU \chi_R(VU) \chi_S(U^+W) = \frac{\delta_{RS}}{\dim(R)} \chi_R(VW) \quad (2.7)$$

$$\int dU \chi_R(UVU^+W) = \frac{\chi_R(V)\chi_R(W)}{\dim(R)} \quad (2.8)$$

The orthogonality equation (2.6) is for totally decomposable (into finite dimensional irreducible unitarian) representations the completeness statement that the irreducible characters form a complete set on the space of continuous class functions, which in turn lie dense in the set of all squared summable classfunctions (see [36] sect.4.3, [46]).

Class functions are functions of group elements which are invariant under conjugation of the group element, i.e. depend only on the conjugacy classes:

$$f(g) = f(hgh^{-1}) = f([g])$$

By the Peter-Weyl theorem [37] each representation of compact groups is totally decomposable into finite dimensional irreducible unitarian representations ([36] section4.3, [46]). This means that there is a “Fourier” expansion for each class function on $U(N)$ and $SU(N)$. By its gauge invariance every summable function of the Wilson loop average can be expressed as a “Fourier” series.

This and the orthogonality of the characters are the two mathematical building blocks to solve QCD_2 exactly. The physical contribution comes from the YM_2 Wilson loop average. This will be done in the following section.

2.4.2 Heatkernel action from continuum limit

We follow [13],sect.2.

Given a cubic lattice with spacing a we choose an axial gauge: $\mathbf{A}_t = 0$. $Z_a(U)$ denotes the “action” associated with the holonomy of an elementary cell.

The correlation function of a joint cell (see figure 2.2)

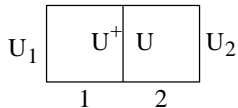


fig. 2.2: joint cell

will be given as

$$Z(U_1 U_2) = \int dU Z_a(U_1 U^+) Z_a(U U_2)$$

Generalising to a block of area A , decomposing Z_a as a function of the Wilson loop (Z_a is a class function then) into a “Fourier-” series with a convenient choice of the “Fourier-” coefficient and using the orthogonality of characters we have:

$$Z_A(V_C) = \sum_R (Z_{a,R})^{A/a^2} \dim(R) \chi_R(V_C)$$

where V_C is the holonomy of the block. A/a^2 is the number of cells in the block (see figure 2.3).

In the small area limit we have to have ($a \rightarrow 0, A/a^2 \rightarrow \infty$):

$$Z_{a,R}(V_C) \rightarrow 1 - \frac{1}{2} a^2 C_2(R) \kappa^2 \quad \text{with} \quad V_C \rightarrow \mathbf{P} \exp \oint_C A_\mu dx^\mu$$

then we have

$$Z_A(V_C) = \sum_R \exp\left(-\frac{A\kappa^2}{2} C_2(R)\right) \dim(R) \chi_R(V_C) \quad (2.9)$$

Because this “heat kernel action” Z_A is invariant under the choice of the lattice spacing a it is the exact solution of QCD_2 . The action is “additive” because of the orthogonality of the characters, i.e. gluing two areas A, A' along an edge (like above) leads to:

$$\int dU Z_A(V U^+) Z_{A'}(U W) = Z_{A+A'}(V W) \quad (2.10)$$

The “gluing” property and the diagrammatic representation of a Riemann surface of genus G with b boundaries (see chapter 5) may be used to evaluate the heat kernel action associated [29]

$$Z(A, G, U_1 \dots U_b) = \sum_R \exp\left(-\frac{A\kappa^2}{2} C_2(R)\right) \dim(R)^{2-2G-b} \prod_{i=1}^b \chi_R(U_i) \quad (2.11)$$

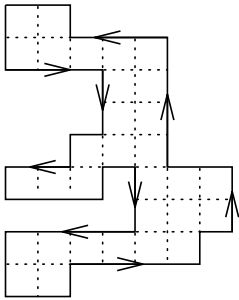


fig. 2.3: block of area A

Chapter 3

Some extracts: From group theory

By making use of the “Fourier-” expansion of the partition function the problem was solved in full generality. The solution, however, is still quite formal: We just transferred the problem to the field of group representations on a direct sum of irreducible invariant spaces.

We have to find explicit expressions for the Casimir eigenvalue on , for the dimension *of* these spaces in terms of a complete, unique classification, such that we can do more than just writing down the solution formally.

The expressions we find should be suitable for a $1/N$ -expansion.

The solutions shall be given, among some background, below.

3.1 The (special) unitary groups

The group structure of $U(N)$, $SU(N)$ is induced by their fundamental representations as matrix groups on a N -dimensional vectorspace V over complex field. These are defined as subgroups of the set of all nonsingular matrices operating on this vectorspace (general linear group) $GL(N)$ by the following constraints.

$$u \in U(N) \text{ or } SU(N): \quad uu^+ = u^+u = 1$$

$$u \in SU(N): \quad \det(u) = 1$$

$GL(N)$ is the largest of the matrix groups for given N . It has four fundamental representations: $\{g\}$ acting on V , $\{g^*\}$ acting on V^* (complex conjugate, $\{g^t\}$ acting on \tilde{V} (transposed, dual space), $\{g^+\}$ acting on \tilde{V}^* (hermitean adjoint) ([30] ch.13).

Because of the condition $u^+ = u^{-1}$ the number of independent fundamental representations comes down to two for $U(N)$. The representations of $SU(N)$ will be dealt with in the last section of this chapter.

The obvious application of the definitions leads to representations of the matrixgroups on tensor product spaces. By definition: any finite dimensional representation is equivalent to a representation on a subspace of $V^{\otimes m} \otimes V^*{}^{\otimes n}$ for some $n, m \in \mathbb{N}$.

The unitary groups are Lie-groups: They are groups with the features of differentiable manifolds. In this sense they are compact. As mentioned before this means that we have an invariant normalisable (Haar-)measure on group space [31] and any representation is completely decomposable into finite dimensional representations.

Subspaces of a carrier space may be “invariant” under group transformations: each element of this subspace is transformed into a linear combination of vectors in this subspace regardless which element is applied. A space which does not

contain any nontrivial invariant subspaces is “irreducible” and so is the associated representation (“IRREP”).

For compact groups each IRREP is equivalent to a finite dimensional unitary representation [36] [46].

The symmetric group S_n of permutations of $(1..n)$ has representations on $V^{\otimes n}$ by permutations of the indices of the tensors in $V^{\otimes n}$. The IRREPs of S_n are wellknown: They are classified by Young tableaux (see eg. [30]chapter5). To a given Young tableau associated is the Young symmetriser Y_n , which generates the IRREPs of S_n on $V^{\otimes n}$ (see figure 3.1).

1	6	10	12
2	7	11	
3	8		
4	9		
5			

fig. 3.1: standard Young diagram

First one has to sum over all permutations which leave the content of the columns unchanged weighted by their sign (antisymmetrising the columns, operator a_{Y_n}), then over all permutations of the rows (symmetrising the rows, operator s_{Y_n}). Then Y_n is given as: $Y_n = s_{Y_n} a_{Y_n}$.

There are different ways to describe a Young tableau, each of which gives a “partition” of n :

- by the number of boxes in each row: $\{n_i\}$, $\sum n_i = n$
- by the number of boxes in each column: $\{c_i\}$, $\sum c_i = n$
- by the number of columns of length i : $\{l_i\}$, $\sum i l_i = n$

By drawing a sketch it is easy to find:

$$c_k = \sum_i \Theta(n_i + 1 - k) \quad \Theta(n) = 1 \text{ if } n > 0, = 0 \text{ if } n \leq 0 \quad (3.1)$$

$$l_j = n_j - n_{j+1} \quad (3.2)$$

All three ways will be used now and then in this work.

We will need the Young symmetriser for two things:

- the Young tableaux do not only classify the IRREPs of S_n , they classify the IRREPs of $GL(N)$ on $V^{\otimes n}$, too. This is roughly speaking, because the operations of S_n commute with those of $GL(N)$ in the maximal possible way ([30] chapter 5 and 13).
- we will use the Young symmetriser to calculate the character of transpositions $\chi_R(T)$ on an irreducible space R . By that we will find a connection between the eigenvalue of the quadratic Casimir $C_2(R)$ and $\chi_R(T)$, and at the same time we will solve the “symmetry-factor” of our string maps. This is, I would say, the most crucial step in the interpretation as it will be outlined in this work.

Finally I want to state Schur’s lemma:

An operator, which commutes with all group operations on an irreducible space, is a multiple of the identity on this space (see eg [30] section 3.4)

This is needed at some stages of the considerations.

3.2 On $C_2(R)$

3.2.1 Lie algebras

This section shall deal with the eigenvalue of the quadratic Casimir $C_2(R)$ and closely related problems. Because, as already stated, $C_2(R)$ has “more” to do

with the Lie algebra of the group than with the group itself it is necessary to give some account of them first.

The elements of a Lie group are, in a neighbourhood of the identity at least, given in terms of a finite number of real parameters: As a Taylor expansion ($g \in G$)

$$\begin{aligned} g(\alpha) &= \mathbf{1} + \sum_{k=1}^r \alpha_k \left(\frac{\partial g}{\partial \alpha_k} \right)_{\alpha_k=0} + O(\alpha^2) \\ &\equiv \mathbf{1} + \sum_{k=1}^r \alpha_k X_k + O(\alpha^2) \end{aligned}$$

The X_k are the generators of infinitesimal group transformations and are a closed set under commutation

$$[X_\sigma, X_\rho] = c_{\sigma\rho}^\tau X_\tau$$

This defines the Lie algebra of the group.

The structure constants $c_{\sigma\rho}^\tau$ are antisymmetric in the lower indices and transform as (2,1)-tensors under changes of the Lie algebra basis.

The scalars

$$I_n = c_{\alpha_1\beta_1}^{\beta_2} c_{\alpha_2\beta_2}^{\beta_3} \dots c_{\alpha_n\beta_n}^{\beta_1} X^{\alpha_1} \dots X^{\alpha_n}$$

are invariant under group transformations. For a Lie algebra of rank l there are l linearly independent such invariants (see eg. [21] [38]).

The operator $I_2 = C_2$ is called the quadratic Casimir and it is quite easy to show that it satisfies:

$$[C_2, X_\rho] = 0 \quad \forall X_\rho \quad [38] \text{ 5.18}$$

For arcwise connected Lie groups this is enough to show that the Casimir is a multiple of the identity on an irreducible invariant space R : $C_2|_R = C_2(R)\mathbf{1}_R$, because then each element can be expressed as pathordered exponential (see eg. [22] p337).

$U(N)$ is connected because each of its elements is diagonalisable and its exponential form can be found almost trivially ([39] p202).¹

For semisimple Lie algebras like $SU(N)$ the tensor $c_{\alpha\beta}^\gamma c_{\gamma\delta}^\alpha$ is nondegenerate (Cartan's criterium). This Killing form / Cartan metric then gives a scalar product on the Lie algebra and the quadratic Casimir reads as $T \cdot T$, $T = (X_1, ..X_r)$.

3.2.2 Cartan-Weyl form

For semisimple Lie algebras there exists a nice basis which makes it easier to find the eigenvalues of C_2 . In a semisimple Lie algebra of rank l there can be found, by definition, l operators in a Lie algebra basis which mutually commute; they form the Cartan subalgebra (the maximal torus) $\{H_i\}$.

The remaining elements of the basis can be arranged in a set $\{E_\alpha\}$ such that:

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

The other relations for the Cartan Weyl form of semisimple Lie algebras are:

$$[H_i, H_k] = 0 \quad [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}$$

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i$$

The vectors E_α are labeled by l numbers α_i . The vector $(\alpha_1, ..\alpha_l) \equiv \alpha$ lies in the “root” space. Root vectors are called positive if the first nonzero component is positive. A basis of the root space only containing positive roots, consists of “simple” roots.

¹ for othogonal groups the discrete quotient group which carries from one connected component to the others has diagonal representations. I do not know whether this is always the case. [48]sect.1.2

The indices in root space are raised and lowered by a metric induced by the Killing form, $g_{\beta\delta} = c_{\alpha\beta}^\gamma c_{\gamma\delta}^\alpha$, which is normalised such that: $g_{\alpha,-\alpha} = 1$ i.e.:

$$g = \begin{pmatrix} g_{ik} & & & & \\ & 0 & 1 & & \\ & 1 & 0 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \\ & & & & \ddots \end{pmatrix}$$

The matrix g_{ik} is a metric on root space ([38] sect.6.5).

This form of a Lie algebra is the one usually used to determine the spectrum of the angular momentum operators J_z, J^2 ([40] 13.1):

$$\begin{aligned} \vec{J} &= (J_x, J_y, J_z) & J_+ &= \frac{1}{\sqrt{2}}(J_x + iJ_y) & J_- &= \frac{1}{\sqrt{2}}(J_x - iJ_y) \\ \Rightarrow [J_z, J_+] &= J_+ & [J_z, J_-] &= -J_- & [J_+, J_-] &= J_z \end{aligned}$$

$J^2 = J_x^2 + J_y^2 + J_z^2$ is the quadratic Casimir of $\text{SU}(2)$.

So we expect that it is possible to give an expression for the eigenvalue of the Casimir in full generality in terms of a basis in this form. Furthermore we expect a connection between the eigenvalues of the Cartan subalgebra and the labeling of irreducible invariant subspaces.

In this context the ‘‘Cartan matrix’’ will be used:

$$A_{ij} \equiv \frac{2 \langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \quad \{\alpha^i\} \text{ simple roots}$$

3.2.3 Weights

Since the l operators H_i are mutually commuting a basis in each representation may be found such that the H_i have the same eigenvectors: $H_i|v\rangle = h_i|v\rangle$.

The vector $h = (h_1, ..h_l)$ is called the “weight vector” of $|v_h\rangle$. For $E_\beta|v\rangle \neq 0$ we have by the commutation relations

$$HE_\beta|v\rangle = (h + \beta)E_\beta|v\rangle$$

This establishes the connection between the root space of the algebra and the eigenspaces of the algebra in representations.

Thereby can be proved that IRREPs can be labeled by “highest weights” $\Lambda = \sum l_i \Lambda_i$, where the $\{\Lambda_i\}$ are the “fundamental weights” defined by

$$\Lambda_j = \sum_{k=1}^l (A^{-1})_{kj} \alpha_k \quad \alpha_k : \text{simple roots} \quad A : \text{Cartan matrix}$$

and the l_i are positive integers ([21] sect.15.3).

In [21], sect.16.7 a discussion which applies to $SU(N)$ is given connecting the highest weight of an irreducible space with a corresponding Young tableau:

$\Lambda = \sum l_i \Lambda_i$ corresponds to a Young tableau with $\{l_i\}$ columns of length $\{i\}$ ([21] p.650).

Let δ be the sum over a half times the sum over all positive roots, i.e.:

$$\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

Then the eigenvalue of the quadratic Casimir in an IRREP labeled by Λ is given as:

$$C_2(R) = \langle \Lambda, \Lambda + 2\delta \rangle \quad [38]15.1, [21]16.1 \quad (3.3)$$

To find the eigenvalue $C_2(R)$ in terms of the variables specifying the Young tableau is now the problem to find a convenient form of the algebra. This problem is solved for quite a while now. Some relevant references and the result are given in the next section.

3.2.4 The eigenvalue

Some reference on the convenient choice of a basis for $U(N)$, $SU(N)$ was given already during the treatment of 't Hooft's $U(N)$ model. [21], App.F,G and [42] contain further reference.

The result of the calculation can be found in various references. The formula used by D. Gross [1] [4] can be found at M.Resnikoff [41], and very easily derived from the expression given eg. in S.Okubo [20]. It is:

$$\begin{aligned}
C_2(R) &= Nn + \sum_i n_i(n_i + 1 - 2i) \quad \text{for } U(N) \\
&\equiv Nn + \tilde{C}(R) \\
&= Nn + \sum_i n_i(n_i + 1 - 2i) - \frac{n^2}{N} \quad \text{for } SU(N)
\end{aligned} \tag{3.4}$$

where the notation of section 3.1 has been used; the last term in the formula for $SU(N)$ “removes the trace”.

In [1] an explicit calculation is given establishing:

$$\sum_j c_j^2 = \sum_i n_i(2i - 1) \quad \text{using } c_k = \sum_i \Theta(n_i + 1 - k) \tag{3.5}$$

leading to (for $SU(N)$)

$$C_2(R) = Nn + \sum_i (n_i^2 - c_i^2) - \frac{n^2}{N} \tag{3.6}$$

This means that in order to find the eigenvalue in terms of the column we have to use ²:

$$C_2(R) = Nn - \sum_i c_i(c_i + 1 - 2i) - \frac{n^2}{N}$$

So $\tilde{C}(R)$ has the following forms:

$$\tilde{C}(R) = \sum_i n_i(n_i + 1 - 2i) = - \sum_i c_i(c_i + 1 - 2i) = \sum_i (n_i^2 - c_i^2) \tag{3.7}$$

² Thereby the eigenvalue of the “dual” R^* , which is R with the rows and columns interchanged, has a different eigenvalue for C_2 . This should not be confused with the eigenvalue of C_2 in its “contragredient” or conjugate representation \bar{R} (cf.sect.3.4), which is the same [45].

3.3 On $\dim(\mathbf{R})$

The next quantity to be evaluated in terms of the Young tableau is the dimension of the irreducible invariant subspace: $\dim(R)$. This is done by a formula given in [43], p.201(5.14) ³:

$$\dim(R) = \frac{D(\lambda_1, \dots, \lambda_N)}{D(N-1, \dots, 1, 0)} = \prod_{i=1}^N \prod_{j=i+1}^N \frac{\lambda_i - \lambda_j}{(N-i)!} \quad (3.8)$$

where $\lambda_i = n_i + N - i$.

A rather nice formula can be found for the relation to the dimension of the associated representation of the symmetric group S_n given by [43]th.7.7p.213:

$$d_R = n! \frac{D(\lambda_1, \dots, \lambda_n)}{\lambda_1! \dots \lambda_n!} \quad (3.9)$$

It can be found in [44], who give original reference for $SU(N)$. The relation is for $SU(N)$:

$$\dim(R) = \frac{d_R}{n!} \prod_{i=1}^r \frac{\lambda_i!}{(N-i)!} \quad (3.10)$$

where r is the number of not empty rows.

Using

$$\frac{(N + n_i - i)!}{(N - i)!} = \prod_{k=1}^{n_i} (N + k - i) = N^{n_i} \prod_{k=1}^{n_i} \left(1 + \frac{k - i}{N}\right)$$

one finds like in [1]:

$$\dim(R) = \frac{d_R N^n}{n!} \prod_v \left(1 + \frac{\Delta_v}{N}\right) \quad (3.11)$$

where v runs over all boxes in the diagram and Δ_v is the difference of column and row index of a box.

³The definition of the difference product can be found at theorem 7.4.B, same reference

By the “hook” formula [37],(4.12):

$$d_R = \frac{n!}{\prod_v(\text{hook length})}$$

(the “hook” length is the number of boxes right and beneath a box plus one)

we see: $d_R = d_{R^*}$ and so:

$$\dim(R^*) = \frac{d_R N^n}{n!} \prod_v \left(1 - \frac{\Delta_v}{N}\right)$$

if one determines Δ_v from R rather than from R^* , its dual (rows and columns interchanged).

3.4 On R

For the “Fourier-” expansion we need all, but not more irreducible representations. In this section a way of labeling all representations in an appropriate way is presented.

According [30] the IRREPs of $U(N)$ are given by a set containing pairs of Young tableaux, one corresponding to representations on $V^{\otimes n}$, the other on $\tilde{V}^{\otimes m}$.

For $SU(N)$ there exists an invariant tensor, the Levi-Cevita symbol/ totally antisymmetric tensor with N indices: ε . Its invariance stems from the condition $\det(u) = 1$ and the connection between the determinant and the Levi-Cevita symbol. It may be used to raise/ lower indices. The raised/ lowered form of an irreducible space will still be irreducible because of the invariance of ε .

The application of ε changes a representation R into its “conjugate” \bar{R} : columns of length c become columns of length $N - c$.

Diagrammatically one gets the conjugate representation \bar{R} of a representation R classified by rows $\{n_i\}$ by drawing a rectangle of $n_1 \times N$ boxes, drawing the columns of R from the bottomline upwards preceeding from right to left (see figure 3.2).

\bar{R} is classified by columns $\{\bar{c}_i\}$, rows $\{\bar{n}_i\}$ given by:

$$\bar{c}_{n_1-i+1} = N - c_i, \bar{n}_i = -n_{N-i+1} + n_1 \quad (\text{see [30]sect.13.4.2}) \quad (3.12)$$

So for $SU(N)$ we have only one Young tableau for a unique classification. Because the column of length N gives the trivial representation of $SU(N)$ these Young tableaux may have column lengths up to $N - 1$ to give a complete, and unique labeling of all IRREPs. This is the result given in [43][47], and matches perfectly well with the correspondence between highest weights and Young tableaux (sect.3.2.3).

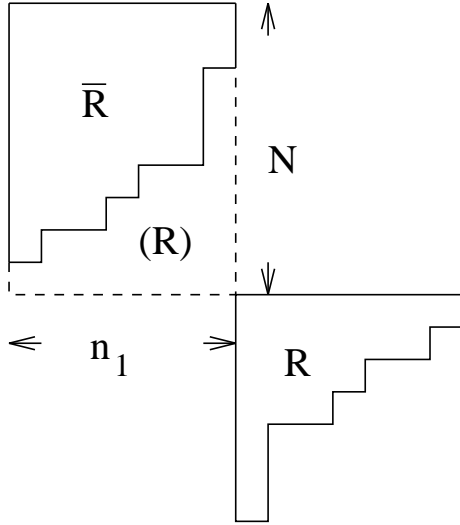


fig. 3.2: conjugate representation

Chapter 4

On the Wilson loop average

Because, usually, Wilson loop averages are not among the objects of physical education, I feel they should be given some consideration for themselves, though, of course, I can not give a “complete” treatment in any way. Now that the terms required for this are introduced the discussion seems to be in order; along the way some remarks of interest to related chapters will be made.

Section 1 will try to relate Wilson loop averages to some fields of relevance to our discussion. In section 2 the experimental proof (Aharonov-Bohm effect), which is often said to have established the importance of gauge potentials on their own (in Quantum Physics), is viewed to prove the importance of Wilson loops. The final section will be used to develop a Hamiltonian formalism on a Hilbert space directly related to the Wilson loops.

4.1 Gauge invariance and lattice theory

In lattice gauge theories the Wilson loop average belongs to the natural variables since holonomies are the essence of the theory. More general: The gauge invariance as fundamental principle of the theory means that its observables, gauge invariant quantities, can not depend significantly on gauge dependent quantities like local

values of a gauge potential. The introduction of a gauge potential as a connection in Yang-Mills theories means that physical variables can not depend on the local value of the gauge field: locally any such connection can be trivialised , i.e. locally each gauge potential is in the same conjugacy class as the identity ([22]p.335). From this point of view it is not counterintuitive that a set of nonlocal, gauge invariant parameters could contain the physical information about a system.

Originally the concept of a loop average was introduced by F.Wegner [63] connected with studies on Ising models with local Z_2 symmetry. It was adapted to nonabelian gauge theories by K.Wilson [7] in order to get a description of confinement.

Confined quarks are one of the “phases” of a quark system one could think of. According to [27](sect.6.1) any order parameter to describe the different phases of a quark system should be nonlocal. Such a quantity is the Wilson loop.

The string tension K , defined as

$$K \equiv \lim_{|C| \rightarrow \infty} -\frac{\ln W(C)}{A(C)}$$

where C is the loop, $|C|$ its length, A its enclosed area and $W(C)$ the Wilson loop average,

enables to distinguish between confined ($K \neq 0$) and not confined phase ($K = 0$).

Wilson comes about “his” loop when he considers a current-current propagator:

$$D_{\mu\nu}(x) = \langle \Omega | T J_\mu(x) J_\nu(0) | \Omega \rangle$$

$|\Omega \rangle$ being the vacuum state, J_μ, J_ν are thought to correspond to e^+e^- -annihilation with resulting quark/antiquark pair production.

So it is diagrammatically: (see figure 4.1)

While calculating the amplitude in the Feynman picture one comes about a weight factor $\exp(i\kappa \oint A ds)$ (cf. section 2) and the possibility of detecting a single

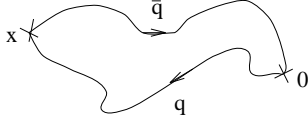


fig. 4.1: quark pair

quark corresponds to contributions from loops with large $q\bar{q}$ - separation. In the strong coupling limit, when the area law holds, large separations are suppressed.

H.Rothe [8] gives a treatment which suggests a simple relation between the ground state energy of a particle/antiparticle pair and the Wilson loop average. The discussion is given for QED (static potential), though.

The two particle state is given as:

$$|\phi_{\alpha\beta}(x, y) \rangle = \bar{\psi}_{\alpha}(x, 0)U(x, 0; y, 0)\psi_{\beta}(y, 0)|\Omega \rangle$$

$$\text{where} \quad U = \exp(i\kappa \int_x^y dz^i A_i(z, t))$$

provides the right gauge transformation behaviour and the connection to the Wilson loop. The following relation arises when considering the time evolution of the state:

$$T \rightarrow \infty : \quad \langle W_C[A] \rangle \longrightarrow \sim \exp(-ET)$$

such that the ground state energy can be calculated as (up to an additive constant):

$$E = - \lim_{T \rightarrow \infty} \frac{1}{T} \ln \langle W_C[A] \rangle$$

4.2 Aharonov- Bohm effect

The Aharonov-Bohm effect is the existence of an observable influence of some current distribution on particles while there is no classical force. This effect was predicted by Y.Aharonov and D.Bohm [64] and observed by H.Boersch et al. [66] and N.Osakabe et al. [67] in the situation depicted in figure 4.2. A jet of

coherent electrons is splitted and guided around a flux tube through an area with no classical field ($E, B = 0$), therefore there is no classical force on the particles.

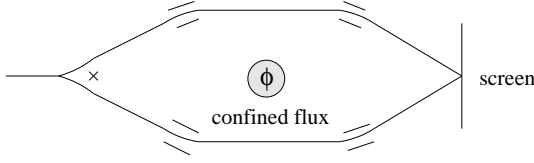


fig. 4.2: experimental situation A-B effect

Nevertheless the particles pick a phase factor due to the gauge field in the area, which is not simply connected. There is no gauge where $A = 0$ everywhere outside the flux tube. (see calculation below)

The experimental proof is then given by the shift of the interference pattern due to a variation of the flux inside the tube.

The evolution of the state along the paths is given by (following [65]):

$$(i\vec{\partial} - \kappa \vec{A} - m)\psi = 0 \quad (4.1)$$

in the case above ($A_0 = 0$)

$$(i\vec{\partial} - \kappa \vec{\gamma} \vec{A} - m)\psi = 0 \quad (4.2)$$

For each *one* path a gradient field $\vec{\nabla} f$ can be found such that

$$\vec{A} = \vec{\nabla} f \quad f = \int^{\vec{r}} d\vec{r} \vec{A}$$

By this gauge transformation

$$(i\vec{\partial} - m)\psi' = 0 \quad \psi' = \exp(-i\kappa \int^{\vec{r}} d\vec{r} \vec{A})\psi$$

Such that the part of the phase difference between the paths which depends on the gauge field A is:

$$\Delta\phi = -\kappa \oint d\vec{r} \vec{A} \quad (4.3)$$

So what is actually observed in the experiments *is* the “Wilson loop average”.

4.3 A 2D Hamiltonian formalism

The great success of field theories is their calculability by *making* the description of interactions local using fields which satisfy certain field equations. A theory based on nonlocal parameters has to have a formalism for calculations. So to conclude this chapter I want to give, following [29], an example of an Hamiltonian formalism based on Wilson loops.

The easiest target space to consider is a cylinder: the periodicity in the x_1 direction allows partial integrations without boundary terms, because we require the field to be well defined. The straight x_0 - direction allows the choice of the axial gauge $A_0 = 0$.

We solve free gauge theory:

The equations of motion are [34](p.568)

$$\partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0$$

Implementing the axial gauge results in the necessity to demand the equation of motion resulting from the variation of A_0 as operator equation on the space of physical states:

(in this gauge $F_{10} = \partial_0 A_1$)

$$\partial_1 F_{10} + [A_1, F_{10}] = 0$$

Taking an infinitesimal, time independent gauge variation δ_ω
(preserves $A_0 = 0$)

$$\delta_\omega A = \partial\omega + [A, \omega]$$

applied to a physical state ψ leads to

$$\begin{aligned} \delta_\omega \psi(A) &= \int dx (\delta_\omega A) \frac{\delta}{\delta A} \psi(A) \\ &= \int dx \omega^a (-\partial \delta_{ab} + A^c f_{abc}) \frac{\delta}{\delta A_b} \psi(A) \end{aligned}$$

leads to the required operator equation:

$$(\partial_1 \frac{\delta}{\delta A_1^a} + [\frac{\delta}{\delta A_1}, A_1]_a) \psi = 0$$

Solutions of the form $\psi[\mathbf{Pexp} \int_0^L dx_1 A_1]$ exist. Invariance under x- independent gauge transformations leads to the conclusion that the Hilbertspace contains functions from $L^2(G)$, the space of squared summable class function on the gauge group G. The scalar product is given by the Haar measure as

$$\langle f_1 | f_2 \rangle = \int dU f_1^*(U) f_2(U)$$

The Hilbertspace is separable by the characters of the group elements in the finite dimensional IRREPs. $|U \rangle$ denotes a state in $L^2(G)$ associated with U, $\langle R | U \rangle$ is the trace of U in the IRREP R. Using the translational symmetry in x_1 we see that “physical” U’s are of the form:

$$U_\Gamma = \mathbf{Pexp} \oint_\Gamma ds A$$

where Γ is a circle at constant time x_0 .

Next we have to find the Hamiltonian. Usually $H = \frac{1}{2}(E^2 + B^2)$, but in two dimensions $B = 0$.

We have

$$\Pi_{A_1} = E = \frac{\delta}{\delta A_1}$$

Taking an orthonormal basis for the Lie algebra $\{T^a\}$, $\frac{\delta}{\delta A_1^a}$ applied to a group element of the form $U_\Gamma = \mathbf{Pexp} \oint_\Gamma ds A$ yields for the trace $\chi_R(U_\Gamma)$

$$\Pi_{A_1^a} \chi_R(U_\Gamma) = \frac{\delta}{\delta A_1^a} \chi_R(U_\Gamma) = \chi_R(T^a U_\Gamma)$$

(using the definition of the path ordered exponential and the rules for functional derivatives)

Therby:

$$\frac{1}{2} (\Pi_{A_1^a})^2 \chi_R(U_\Gamma) = \frac{1}{2} \chi_R(T^a T^a U_\Gamma) = \frac{1}{2} C_2(R) \chi_R(U_\Gamma)$$

$$\Rightarrow H|_R = \frac{1}{2}C_2(R)$$

So the quantum mechanical propagator for two states $|U_1\rangle$, $|U_2\rangle$ (both at $t = 0$) is given by: (cf. figure 4.3)

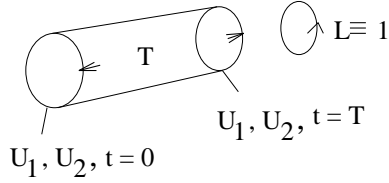


fig. 4.3: quantum mechanical propagator

$$\begin{aligned} Z(U_1, U_2, T) &= \langle U_1 | e^{-TH} | U_2 \rangle \\ &= \langle U_1 | \sum_R | R \rangle \langle R | e^{-TH} \sum_{R'} | R' \rangle \langle R' | | U_2 \rangle \\ &= \sum_R \langle U_1 | R \rangle e^{-\frac{1}{2}T C_2(R)} \langle R | U_2 \rangle \\ &= \sum_R \chi_R(U_1^+) \chi_R(U_2) \exp\left(-\frac{1}{2}T C_2(R)\right) \end{aligned}$$

The possibility to make an infinitesimal region flat means that the holonomies around such a region are the identity then and the space of states on such a target space has the form:

$$\psi(U) = \delta(U)$$

Using both results and gluing an infinitesimal disc to the cylinder we get for a plaquette of area A :

$$Z(A, U) = \sum_R \dim(R) \chi_R(U) \exp\left(-\frac{1}{2}A C_2(R)\right)$$

J.Baez and W.Taylor [55] develop a Hamiltonian formalism using “loop variables” (cf. chapter 8).

Chapter 5

Some extracts: About Riemann Surfaces

By reducing a geometrical object to a topologically equivalent triangulation it is possible to transfer the problem of finding topological characteristics of this object to a field where algebraic methods, from group theory in particular, can be applied.

Homology, homotopy and monodromy groups are such methods. Their concept is to assign elements $\{g_i\}$ of known groups (here: \mathbf{Z} , and S_n respectively) to geometrically “distinct” objects $\{x_i\}$. A group structure is induced for these objects by their partners: $x_i^{g_j} x_i^{g_k} = x_i^{g_j g_k}$. In this sense the group structure is “formal” .

The “geometrically distinct” objects are found when one specifies subsets of the triangulations and, possibly, identifies their elements by equivalence relations. The representatives of such equivalence classes (“generators”) are then the objects that receive a partner from the group.

In the cases under consideration the number of generators is always finite and the structure of the groups is quite simple.

In these concepts do fit the “branched coverings”, too. They are introduced

to have strings in the theory. Their classifying group, the monodromy group, is a subgroup of S_n and establishes the connection to the theory of the gauge group.

5.1 Homology, Homotopy

Given a set of “generators” $\{x_i\}$ we introduce a group structure by $x_i^{g_j} x_i^{g_k} = x_i^{g_j g_k}$ (we take $g_j \in \mathbf{Z}$).

Because a priori nothing is known about a possible Abelian character we have to assume that every element in this finitely generated group can be written as a “word” using the generators as “letters”:

$$w = x_{j_1}^{i_1} \dots x_{j_n}^{i_n}$$

for some finite positive number n .

A word is called “reduced” if $\forall j : i_j \neq 0, x_j \neq x_{j+1}$; the empty word ($\forall j : i_j = 0$) is $\mathbf{1}$.

The product of two words is obviously:

$$vw = \{x_{j_1}^{i_1} \dots x_{j_n}^{i_n}\} \{x_{l_1}^{k_1} \dots x_{l_m}^{k_m}\} = x_{j_1}^{i_1} \dots x_{j_n}^{i_n} x_{l_1}^{k_1} \dots x_{l_m}^{k_m}$$

A group G generated in this way with no more constraints is “free”. Abelian groups satisfy:

$$a_i a_j = a_j a_i \Leftrightarrow a_i a_j a_i^{-1} a_j^{-1} = \mathbf{1} \quad \forall i, j$$

and words in those groups can be furtherly reduced using these relations.

(The homology group is abelian)

So in general a finitely generated group is defined by a set of generators $\{x_i\}$, the inducing group and a set of relations $\{r_j\}$, under which words are identified (i.e. reduced to the same reduced form). The group then consists of these reduced words. But for our purpose the form $G = (\{x_i\}, \{r_j\})$ is the appropriate one ([22]sect.4.4).

On homology:

Taking a triangulation K of a surface M one has a lot of triangles (equally well polygons). One may build surfaces or lines with them by taking connected subsets of the triangulation. Taking connected edges of the triangles we get “1-chains”, the 1-chains without ends are called “1-cycles”. They generate the 1-cycle group $Z_1(K)$.

The boundaries of connected subsurfaces of the triangulation generate the the 1-boundary group $B_1(K)$.

Identifying all 1-cycles which together are a boundary gives the first homology group

$$H_1(K) = Z_1(K)/B_1(K)$$

The homology groups of two triangulations K, K' of the same surface M are isomorphic; the homology group of M may be defined and taken to be the one of some triangulation K :

$$H_1(M) \equiv H_1(K) \cong H_1(K') \quad [22]\text{def.3.12,13,16}$$

For compact orientable surfaces of genus g one can choose $2g$ generators in the way sketched in figure 5.1.

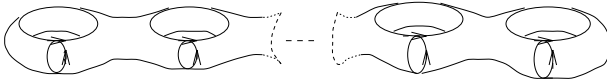


fig. 5.1: homology generators

After deformation of the surface the generators may be chosen such that they all have a common basepoint P . The generators have an orientation. “Cutting” along the generators and identifying according to the orientation of corresponding edges leads to a representation of a $2g$ -surface as a $4g$ -gon ([22]fig.2.6,12) (see figure 5.2, where the assigned group elements stand for the way one has to travel over the generators for a closed path along the $4g$ -gon).

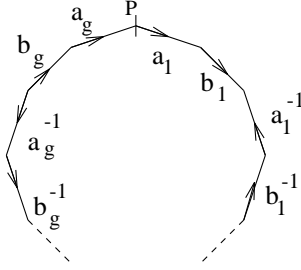


fig. 5.2: representation as a $4g$ -gon

We leave the treatment of boundaries to the end of the paragraph on homotopy.

On homotopy:

The objects leading to the homotopy group are “loops”; these are continuous “paths” $\alpha : [0, 1] \rightarrow M$ with $\alpha(0) = \alpha(1)$.

The product of paths with $\alpha(1) = \beta(0)$ is $\alpha * \beta : [0, 1] \rightarrow M$

$$\alpha * \beta(s) = \alpha(2s) \quad \text{if } s \in [0, \frac{1}{2}] \quad , \quad = \beta(2s - 1) \quad \text{if } s \in [\frac{1}{2}, 1]$$

The same definition applies to loops.

Two loops based at $x_0 \in M$ are said to be equivalent (“homotopic”) if there exists a continuous map $F : [0, 1] \times [0, 1] \rightarrow M$ with

$$F(s, 0) = \alpha(s) \quad F(s, 1) = \beta(s) \quad F(0, t) = F(1, t) = x_0 \quad \forall t \in [0, 1]$$

The product and the equivalence relation “homotopy” define the first homotopy group $\Pi_1(M)$ (“fundamental group”) ([22]def.4.6, lemma4.7, th.4.8)

The generators given for the homology group of the torodi of genus g generate the homotopy group, too. In terms of these generators:

$$\Pi_1(M) = (\{a_i, b_i\}_{i \in g}; \prod_i a_i b_i a_i^{-1} b_i^{-1} = \mathbf{1}) \quad [22](4.26)$$

The classification of torodi described above is equivalent to the classification of a genus g surface as a two sphere with g handles [49]1.3.7. In both formalisms orientable surfaces with m boundaries are included by making m punctures in the sphere, the 4g-gon respectively.

The new generators are included by sending loops from P around the punctures([49]1.3.9). Viewed on the punctured sphere the formalism is convincing and the homotopy group is given by:

$$\Pi_1(M \setminus \{q_j\}_{j \in b}) = (\{a_i, b_i, c_j\}_{i \in g, j \in b}; \prod_j c_j \prod_i a_i b_i a_i^{-1} b_i^{-1} = \mathbf{1})$$

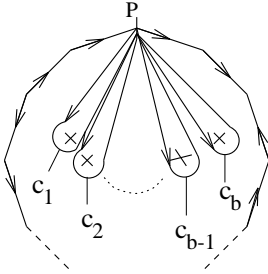


fig. 5.3: representation of b boundaries

5.2 Branched coverings

The mathematical interest in Riemann surfaces stems from the fact that the concept of a “complex structure” like in the complex plane can be introduced on them. Because of this Riemann surfaces are used to do complex analysis and a lot of treatments on Riemann surfaces involve heavy complex analysis.

The concept of “complex structure” on manifolds is local: basically only an operator J with $J^2 = -1$ has to exist on the manifold ([22]p.275). But the full strength of complex analysis can only be achieved on manifolds which admit such

a structure globally. These manifolds are “complex”, i.e. they can be given by local charts $\phi : M \rightarrow \mathbf{C}$.

The concept of “holomorphicity” on \mathbf{C} is local: the Cauchy-Riemann equations are local. This term is introduced for maps between complex manifolds M, N (Riemann surfaces are complex): functions $f : M \rightarrow N$ are holomorphic, if, in terms of local charts $\phi(\psi)$ on M (N), $\psi \circ f \circ \phi$ is holomorphic in \mathbf{C} .

After the choice of a specific complex structure on N there is only one complex structure on M to make a differentiable map $f : M \rightarrow N$ holomorphic: The pullback $f^*(J)$ [51].

Because the arrangements described above prove to be useful Riemann surfaces are defined to be one (complex) dimensional analytic manifolds ([50]p.9). Usually the requirement of connectedness is added. For the results needed in our context this restriction is not necessary and, because a partition function has something to do with disconnected diagrams, not desirable either. The mathematical statements needed and given below generalise to disconnected Riemann surfaces.

Holomorphic maps f between Riemann surfaces M, N have nice features, eg. $f : M \rightarrow N$ holomorphic, M compact, then f is either constant or surjective. In the latter case N is also compact ([50]I.1.6).

“Covers” are locally diffeomorphic, surjective mappings between manifolds ([22]def.4.48).

Such surjective maps have definite “winding numbers”: A nonconstant holomorphic cover $f : M \rightarrow N$ takes on the value $f(P)$ n times “for all” $x \in M$. (for some finite positive number n) ([50]I.1.6).

Consider $N = \mathbf{C} \cup \{\infty\}$, the Riemann sphere, and the map $w = z^2$. w runs two times around \mathbf{C} ; the map \sqrt{w} covers \mathbf{C} twice; there are two fixed points: $0, \infty$ (“branch” points).

w may be cut into two “sheets” along a great circle between the two branch points 0 and ∞ . The edges of the sheets are identified such that the result is w

again. The covering from w onto \mathbf{C} is then locally diffeomorphic.

Inverse images of paths around or through 0 (or ∞)(under the covering map) may start on one sheet and end after identification at one of the edges on another: branch points are, in this sense, “tubes” lifting from one sheet to another.

In the neighbourhood of the branch points 0 and ∞ the covering space w looks in suitable local coordinates like z^2 ; two is here the number of sheets. (example from [49]1.1.1)

All these statements generalise for coverings M (Riemann surface) of a Riemann surface N : $f : M \rightarrow N$ a holomorphic cover.

f has a finite number of “branch” points P on N (the branch locus S) in whose neighbourhoods there exist local coordinates such that the covering locally looks like z^n , where n is constant everywhere and is the winding number. Each branch point gets a branch index $b(P) = n - 1$. The branching number B is defined to be

$$\sum_{P \in S} b(P) \equiv B$$

The definitions are closely related to the Riemann-Hurwitz relation and its derivation

The Euler-Poincaré characteristic for compact surfaces is given as

$$\chi = V - E + F \quad (\text{see chapter 1})$$

For orientable surfaces: $\chi = 2 - 2g$, where g is the genus. The Euler-Poincaré characteristic has the same definition for connected as well as for disconnected surfaces and may serve as the defining quantity for the “more general” genus using the relation above. The genus g of such disconnected sets of surfaces may have any value between $-\infty$ and $+\infty$.

The Euler-Poincaré characteristic χ is all that is needed to prove the Riemann-Hurwitz relation ([50]I.2.6,I.2.7):

M , N Riemann surfaces of genus g , G ; $f : M \rightarrow N$ a holomorphic map of degree n , branching number B ; then

$$2(g - 1) = 2n(G - 1) + B \quad \text{remark: } B \text{ is always even} \quad (5.1)$$

Closely related is Kneser's formula, which is not needed in the following treatment ([52]p.73):

$$2(g - 1) \geq 2n(G - 1) \quad (5.2)$$

5.3 Monodromy group

As in the example of section 5.2 we imagine the covering space M to be cut along great circles between the branch points into n sheets. They are labeled from 1 to n and the edges are identified; because outside the branch points the covering map is locally diffeomorphic, and even though partially disconnected, the sheets of one *connected* component of the covering have, after the identifications, to be *connected* again.

An identification is a permutation Π_q of $(1...n)$, which tells where a circle around the branch point q , starting after one identification in the sheet i , carries with the next identification: onto the sheet $\Pi_q(i)$.

These permutations, associated to the branch points, form a subgroup of the symmetric group, the “monodromy group” ([49]p.57). Considering branched coverings of the sphere with m branchpoints one finds the relation of the monodromy group.

Choose a point P on some sheet i outside every branch point, cut the sheets on all levels along great circles from P to every branch point. A circle around P (see figure 5.4) ends on the sheet

$$\Pi_m \Pi_{m-1} \dots \Pi_2 \Pi_1(i)$$

but also has to end on the sheet i , because P is not a branch point. So we have:

$$\Pi_m \Pi_{m-1} \dots \Pi_2 \Pi_1 = \mathbf{1} \quad (5.3)$$

This looks exactly like the relation defining the fundamental group $\Pi_1(S^2 \setminus S)$.

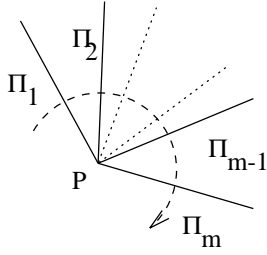


fig. 5.4: consistency relation

And, indeed, the definition above induces an homomorphism

$$\Pi_1(S^2 \setminus S, P) \rightarrow S_n$$

and furthermore: the treatment generalises by the same procedure, only slightly more complicated by the identifications along the generators for the holonomy group, for any Riemann surface, such that an homomorphism: $\Pi_1(N \setminus S, P) \rightarrow S_n$ is given. (P is the basepoint of the groups.)

There exists a theorem which makes these homomorphisms most relevant for our discussion. For this first two definitions:

- Two branched covers $f_1, f_2 : M \rightarrow N$ are said to be *equivalent* if there exists diffeomorphism $\phi : M \rightarrow M$ such that $f_1 \circ \phi = f_2$
- Two homomorphisms $\psi_1, \psi_2 : \Pi_1(N \setminus S, P) \rightarrow S_n$ are said to be *equivalent* if they differ only by an inner automorphism (“similarity transformation”) of S_n , i.e. $\exists g \in S_n : \psi_1 = g\psi_2g^{-1}$

The theorem is [54] [53] [29]:

S a finite subset of a Riemann surface N , n a positive integer, $P \in N$, $P \notin S$.

There is a one to one correspondence between equivalence classes of homomorphisms $\psi : \Pi_1(N \setminus S, P) \rightarrow S_n$ and equivalence classes of n -fold branched coverings with all elements of S as branch points.

This theorem will be used to calculate a “symmetryfactor” $|S_n|$ connected with a branched covering, which is already determined by the QCD_2 partition function, but can be reproduced consistently by this theorem within the string picture.¹

¹The argumentation in [4] proceeds formally a little bit differently; the connection is given in [29]sect.5.1.2

Chapter 6

Interpretation: One chiral sector

6.1 Introduction

A QFT-perturbation theory is usually a set of Feynman-rules which define a connection between a set of topological objects, the diagrams, and probability amplitudes for physical processes.

The Feynman diagrams are “topologically inequivalent”, i.e. certain geometrical objects, and associating things like vertices, propagators etc. means that they are embedded in spacetime; only the topological features of the embedded form are relevant to the theory.

String perturbation theory, in these respects, follows the same procedure. “Embedding” strings in a two dimensional compact manifold makes things special. To make these maps surjective they only have to be holomorphic. A consistent interpretation using branched coverings proves to be possible.

Orientations, which one may assign to orientable diagrams, strings or the skeletons of common QFT, have no meaning to the amplitudes of free gauge theory. Meaningful orientations are only associated to fermion lines, which are, following 't Hooft's work, considered to be suppressed in $N \rightarrow \infty$.

In the perturbative expansion of the partition function related to actions such

as the Nambu-Goto action, one could equally well sum over both orientations and all strings, if these admit orientations: This would only result in a new normalisation of the partition function.

In chapter 7 and 8 some argumentation will be given for introducing two sets of strings with “opposite” orientation. In this context the discussion given in this chapter applies to each *one* set of surfaces. Therefore the term “chiral” partition function or “chiral” sum.

The partition function contains four terms which have to be understood in terms of strings:

$$\begin{aligned} Z(G, \lambda A, N) &= \sum_R \dim(R)^{2-2G} \exp\left(-\frac{\lambda A C_2(R)}{2N}\right) \\ &= \sum_R \dim(R)^{2-2G} \exp\left(-\frac{\lambda A n}{2}\right) \exp\left(-\frac{\lambda A}{2N} \tilde{C}(R)\right) \exp\left(\frac{\lambda A n^2}{2N^2}\right) \end{aligned}$$

where $\lambda = \kappa^2 N$ chosen to take the limit motivated by 't Hooft [2] $N \rightarrow \infty \wedge \lambda = \kappa^2 N = \text{const.}$ (cf. section 1.2.2).

The terms of the $1/N$ -expansion related to the first two exponentials will be understood by means of branched coverings. The first one already looks like the area of a string which winds n times around the target space of area A (sections 3 & 4).

The last term give rise to the introduction of “tubes” and “collapsed” (“infinitesimal”) handles (section 5).

6.2 Chiral 1/N- expansion

In the recent chapters the following formulae were achieved:

$$Z = \sum_R \dim(R)^{2-2G} \exp\left(-\frac{\lambda A}{2N} C_2(R)\right)$$

$$\dim(R) = \frac{d_R N^n}{n!} \prod_v \left(1 + \frac{\Delta_v}{N}\right) \approx \frac{d_R N^n}{n!} + O(N^{n-1})$$

$$C_2(R) = Nn + \tilde{C}(R) - \frac{n^2}{N}$$

Using these and following the discussion of the previous section leads to the expansion below:

$$Z = \sum_{n=0}^{\infty} \sum'_{R \in Y_n} \dim(R)^{2-2G} \exp\left(-\frac{\lambda A}{2N} C_2(R)\right)$$

$$\approx \sum_{n=0}^{\infty} \sum'_{R \in Y_n} \exp\left(-\frac{\lambda A n}{2}\right) \times$$

$$\sum_{i=0}^{\infty} \left(\frac{(-\lambda A \tilde{C}(R))^i}{2^i i!} N^{n(2-2G)-i} + O(N^{n(2-2G)-i-1}) \right) \quad (6.1)$$

The prime on the summation indicates that only the Young tableaux with columns of length less or equal $N - 1$ are to be summed over.

The $O(\cdot)$ indicates the subleading contributions from the last term in $C_2(R)$ as well as contributions from the dimension (for $G \neq 1$).

The branching number in the Riemann-Hurwitz relation, whose look-a-like can be found above, is always even, while the summation index i may take any value in the limits. But although the Riemann-Hurwitz relation applies only to half of the sum we have no serious problem with that: The formalism will be generalised such that we can include odd “branching” numbers naturally.

6.3 Strings and branched coverings

Taking λ as the string tension and $1/N$ as string coupling we know quite a lot about a string expansion of closed, orientable strings:

λ should always appear together with the area A and a string of genus g contains $2g - 2$ vertices, so have to have a factor N^{2-2g} in the expansion (see figure 6.1).



fig. 6.1: holes and vertices

Because the perturbation theory of a partition function contains disconnected diagrams, the series in g has to range from $-\infty$ to $+\infty$ (with $\chi \equiv 2 - 2g$) (cf. section 5.2).

Observing the Riemann-Hurwitz look-a-like in the expansion we are tempted to write down:

$$Z(G, \lambda A, N) = \sum_{g=-\infty}^{+\infty} \sum_n \sum_i \zeta_{g,G}^{ni} \exp\left(-\frac{n\lambda A}{2}\right) (\lambda A)^i N^{2-2g}$$

$$\text{with } \zeta_{g,G}^{ni} = \sum_R \left(\frac{n!}{d_R}\right)^{2G-2} \frac{1}{i!} \left(\frac{\tilde{C}(R)}{2}\right)^i$$

In the interpretation n will be the winding number and i will be the “branching” number of a covering.

As usual in perturbation theory there will be a “symmetry factor” associated with a diagram. This will turn out to be essentially the coefficient $\zeta_{g,G}^{ni}$: $i! \zeta_{g,G}^{ni} =$ symmetry factor.

It would seem to be intuitive to weight the coverings with winding number n and “branching” number i , which are the labels we are summing over, with the number of *different* coverings with these coefficients. This approach will give the required term for the interpretation.

Usually symmetry factors are determined such that the theory becomes unitarian. Because QCD_2 is already a gauge field theory reproducing the right terms by defining a symmetry factor should yield a correct “string” theory.

6.4 Calculating the symmetryfactor

For the calculation of the symmetry factor we will count, as described in chapter 5, the number of distinct homomorphisms $\#[H_\nu]$:

$$H_\nu : \Pi_1(N \setminus \{q_i\}) \rightarrow S_n$$

connected with a covering map ν . The set of covering maps of a target space with genus G with winding number n and “branching” number i will be denoted as $\Sigma(G, n, i)$.

For the purposes of counting the distinct homomorphisms the branch points with branching index $b(q_i) = n - 1$ may be decomposed into $n - 1$, what I want to call, “decomposition” points:

To each branch point q_j is associated a permutation Π_j of $(1...n)$. Now: every permutation may be decomposed into at most $n - 1$ transpositions. It is possible, obviously, to define some algorithm such that the decomposition becomes unique.

The permutation is then a sequence of transpositions. When the homomorphism was defined all that was required was to know where a path starting on some sheet i , going around some branch point is heading with the next identification. This involves one of the transpositions only. So we may associate to each transposition one point, which connects two sheets only and may be located anywhere on the targetspace. In spite of the similarity to branch points I will call these points “decomposition” points to mark their origin and their character.

This decomposition is helpful to calculate the symmetry factor and, thereby, to find the correspondence to the chiral sum, and it is necessary to generalise the formalism to odd “branching” numbers.

The decomposition points may be distributed in $\frac{(\lambda A)^i}{i!}$ ways: each decomposition point has the whole area of the target space as possible location, but the labeling of the points is unimportant. This gives already one of the required terms.

If one sums over i transpositions and $2G$ general permutations with a delta-function sensitive to the defining relation of the monodromy group, which is invariant under conjugation of its argument, one will get $n!$ the number of distinct homomorphisms, because of the freedom of conjugation. Summing with a constant weight $\frac{1}{n!}$ should give the right result. the actual evaluation of this sum has, of course, to do with the representation theory of the symmetric group.

For the symmetric group the following equations are valid:

$$\frac{1}{n!} \sum_R \chi_R(\mathbf{1}) \chi_R(\rho) = \delta(\rho) \quad [30] \text{ (3.6-3)} \quad (6.2)$$

$$\sum_R \chi_R(\rho) D_R(\rho^{-1}) = \frac{n!}{d_R} \mathbf{1}_R \quad [30] \text{ th.3.5} \quad (6.3)$$

$$\sum_{\sigma \in S_n} D_R(\sigma \rho \sigma^{-1}) = \frac{n!}{d_R} \chi_R(\rho) \mathbf{1}_R \quad [30] \text{ cf. proof of lemma 3.7} \quad (6.4)$$

$$\sum_{\tau \in T} D_R(\tau) = \frac{1}{d_R} \frac{n(n-1)}{2} \chi_R(\tau) \mathbf{1}_R \quad [30] \text{ lemma to th.3.7} \quad (6.5)$$

where T is the conjugacy class of transpositions and $D_R(\tau)$ is the representative of τ .

The sum described before reads as

$$\sum_{\tau_1 \dots \tau_i \in T} \sum_{s_1, t_1 \dots s_G, t_G \in S_n} \frac{1}{n!} \delta(\tau_1 \dots \tau_i \prod_{j=1}^G s_j t_j s_j^{-1} t_j^{-1}) \quad (6.6)$$

From (6.4) we have:

$$\sum_{s, t \in S_n} D_R(sts^{-1}t^{-1}) = \left(\frac{n!}{d_R}\right)^2 \mathbf{1}_R \quad (6.7)$$

Rewriting the delta function with (6.2), writing the character as trace over a product of matrices and using (6.5) and (6.7) leads to:

$$\begin{aligned} & \sum_R \left(\frac{1}{n!}\right)^2 d_R \left(\frac{n!}{d_R}\right)^{2G} \left(\frac{n(n-1)\chi_R(T)}{2d_R}\right)^i \text{tr} \mathbf{1}_R \\ &= \sum_R \left(\frac{n!}{d_R}\right)^{2G-2} \left(\frac{n(n-1)\chi_R(T)}{2d_R}\right)^i \end{aligned}$$

What remains to be done is the calculation of the character of transpositions in an arbitrary IRREP R . One possible procedure is the following:

The operator

$$\tilde{T} = \sum_{\tau \in T} \tau$$

commutes with S_n by the rearrangement lemma, since every permutation is decomposable into transpositions. According to Schur's lemma \tilde{T} has to be a multiple of the identity on any IRREP of S_n .

To find the factor it is sufficient to calculate the matrixelement $\langle v | \tilde{T} | v \rangle$ for some $|v\rangle \in R$. For this calculation we need the properties of the Young symmetriser Y_R . I will give illustrations for one of the first nontrivial examples and an argumentation that applies to the general case.

Given a Young tableau one may get a vector by first labeling the boxes from one to n , which then label the components of a tensor in $V^{\otimes n}$. The choice here is a “standard” Young diagram ([30]p.66) (see figure 6.2).

1	4
2	
3	

fig. 6.2: example: standard Young diagram

A vector in R can then be found by choosing an appropriate vector in $V^{\otimes n}$. The coordinates (in terms of a basis in V) should be different for the indices in one column, because the Young symmetriser antisymmetrises the columns, eg. $|1, 2, 3; 1\rangle$.

Y_R then generates $|v\rangle$ in two steps: First the columns are antisymmetrised, i.e.:

$$a_{Y_n} = (123)(4) + (231)(4) + (312)(4)$$

$$-(213)(4) - (132)(4) - (321)(4)$$

$$a_{Y_n} |1, 2, 3; 1\rangle = |1, 2, 3; 1\rangle + |2, 3, 1; 1\rangle + |3, 1, 2; 1\rangle$$

$$-|2, 1, 3; 1\rangle - |3, 2, 1; 1\rangle - |1, 3, 2; 1\rangle$$

Transpositions are labeled by the “boxes they transpose”. The result of the antisymmetrisation is a vector which is antisymmetric under transpositions within a column. This antisymmetry is indicated by links between the antisymmetrised boxes (see figure 6.3).

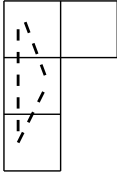


fig. 6.3: antisymmetrised vector

Next the rows are symmetrised:

$$s_{Y_n} = (14)(2)(3) + (41)(2)(3)$$

$|v\rangle$ is given by

$$s_{Y_n} a_{Y_n} |1, 2, 3; 1\rangle = |v\rangle$$

diagrammatically:(see figure 6.4)

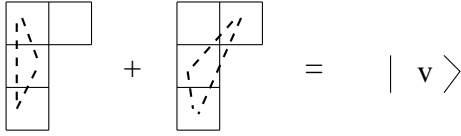


fig. 6.4: vector in IRREP

In the general case the resulting vector is a sum of $n_1! \dots n_l!$ sums, each of which is symmetric under transpositions within rows and antisymmetric under a different set of $\sum \frac{c_i(c_i-1)}{2}$ transpositions, as indicated in the diagram.

In the following $|v\rangle$ is assumed to be normalised, i.e. $\langle v|v\rangle = 1$.

To evaluate $\langle v|\tilde{T}|v\rangle$ we have to consider three cases:

- The $\sum \frac{n_i(n_i-1)}{2}$ transpositions within rows produce each a number +1 by the rearrangement lemma since s_{Y_n} is the sum of *all* such transpositions.
- Each term in $|v\rangle$ has $\sum \frac{c_i(c_i-1)}{2}$ “vertical” transpositions it is antisymmetric under. Summing over all vertical transpositions and gathering only the terms with a definite antisymmetric behaviour leads to $-\sum \frac{c_i(c_i-1)}{2}|v\rangle$ such that the contribution from this side is:

$$-\sum \frac{c_i(c_i-1)}{2}$$

- The action of vertical transpositions on terms with no definite symmetry makes the following contribution:

Because there is no definite symmetry/ antisymmetry the transposition follows neither a row nor a link. Necessarily at least one of the links ends up

being “horizontal”, i.e. lying in a row (see figure 6.5). Using this antisymmetry the contribution to the matrix element is zero (see figure 6.6).

$$(4\ 3) \begin{array}{|c|c|} \hline \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} & \\ \hline \end{array} = \begin{array}{|c|c|} \hline \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} & \begin{array}{c} \diagup \\ \diagdown \end{array} \\ \hline \end{array}$$

fig. 6.5: “horizontal links”

$$\begin{aligned} \langle v \mid (4\ 3) \begin{array}{|c|c|} \hline \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} & \\ \hline \end{array} &= \langle v \mid (4\ 1)^2 \begin{array}{|c|c|} \hline \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} & \begin{array}{c} \diagup \\ \diagdown \end{array} \\ \hline \end{array} \\ &= \langle v \mid (-1) \begin{array}{|c|c|} \hline \begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} & \begin{array}{c} \diagup \\ \diagdown \end{array} \\ \hline \end{array} = 0 \end{aligned}$$

fig. 6.6: vanishing contributions

Thereby we have found:

$$\begin{aligned} \langle v \mid \tilde{T} \mid v \rangle &= \sum_i \frac{n_i(n_i - 1)}{2} - \sum_i \frac{c_i(c_i - 1)}{2} \\ &= \frac{1}{2} \sum_i n_i^2 - c_i^2 \end{aligned} \tag{6.8}$$

This leads to

$$\text{tr}(\tilde{T}) = \frac{n(n - 1)}{2} \chi_R(T) = \frac{\tilde{C}(R)d_R}{2} \tag{6.9}$$

This proves: $i! \zeta_{gG}^{ni}$ is given by our symmetry factor. For the sum over coverings the symmetry factor will be written as:

$$\sum_{\nu \in \Sigma(G, n, i)} \frac{1}{|S_\nu|}$$

6.5 “Tubes” and “contracted handles”

What is still possible to do within this chapter is to interpret the contribution from the last term in $C_2(R)$: $-\frac{n^2}{N}$. The ideas for this come from J.Minahan [3].

We rewrite the argument of the exponential as:

$$\frac{n\lambda A}{2N^2} + \frac{n(n-1)}{2} \frac{\lambda A}{N^2}$$

The first term is interpreted as an “infinitesimal handle” ([3]) or “collapsed” handle attached to the covering space and mapped entirely on a single point in the target space. This object increases the genus of the covering space by one (referring to the square of N), gives a factor of $n\lambda A$ for the choice of its position and a half for the indistinguishability of the two ends of the handle.

The second term is viewed as a “tube” connecting two sheets of the covering space: This increases the genus by one (due to the “hole” created by the tube) and for the choice of its position we have $\frac{n(n-1)}{2}\lambda A$ (a tube relates to a transposition in this sense).

These tubes are “equivalent” to the coincidence of two decomposition points. They are orientation preserving, i.e. they do not interfere with the choice of some orientation for the covering space. Increasing the number of tubes/ handles and counting in the right combinatorial way leads to the exponentiation of the term. (Tubes and handles are “local”.)

Chapter 7

Interpretation: Nonchiral sum

So far there has been no reason for using the orientability of the covers in an explicit way: All terms could be interpreted consistently without mentioning any orientation.

The “chiral” theory is the exact solution of the partition function; but as the following examination will show it is not what we are going for: in the $N \rightarrow \infty$ limit the series does not give the leading contributions first. A more useful series will be developed by an approximate rearrangement of the chiral sum yielding the “nonchiral” sum. The procedure introduces a double sum over representations (two chiral “sectors”) instead of the single one we had before and an “additional” term in the exponential weight which couples the two sectors, which are else (approximately) independent. The interpretation of this coupling term (the “orientation reversing tube”) makes use of two “different” orientations on the two covers each belonging to one chiral sector.

7.1 Leading “composite” representations

In the $N \rightarrow \infty$ limit the weight in the sum is the exponential: if the argument in the exponential is of order N the contribution is damped as $\exp(-O(N))$ [29].

While if the argument is of order $1/N$ the main contribution comes from the zeroth order term. So it seems the most “interesting” contributions to the theory come from representations which produce arguments of order N^0 .

So it might be instructive to examine the weight once again:

$$\exp(-\frac{\lambda A}{2N}C_2(R)) = \exp(-\frac{\lambda A n}{2}) \exp(-\frac{\lambda A}{2N}\tilde{C}(R)) \exp(\frac{\lambda A n^2}{N^2})$$

The first and the third term tell that $n \approx O(N)$ is most interesting. The second term is the one which establishes the key of the interpretation, the connection to “branched coverings”. To make this one interesting we have to have $\tilde{C}(R) \approx O(N)$.

$\tilde{C}(R)$ is essentially given by the lengths of the columns compared with the length of rows. The columns are bounded by $N - 1$. In a constellation where the columns have length of $O(N)$ and where we do not have too many columns in the diagram we will have good contribution from all terms. Those are easily found as “composite representations”.

We take two “small” representations R, S . “Small” refers to the comparison of the parameters specifying the representation with the gauge group parameter N .

We build the conjugate representation \bar{S} of S and combine the rows of R and \bar{S} . The result is the composite representation $T = \bar{S}R$ (see figure 7.1).

S may have columns of length $\{\tilde{c}_i\}, \sum \tilde{c}_i = \tilde{n}$, R of $\{c_i\}, \sum c_i = n$ then the columns of T are given as

$$\begin{aligned} \kappa_i &= N - \tilde{c}_{\tilde{n}_1+1-i} \quad \text{for } i \leq \tilde{n}_1 \\ &= c_{i-\tilde{n}_1} \quad \text{for } i > \tilde{n}_1 \end{aligned} \tag{7.1}$$

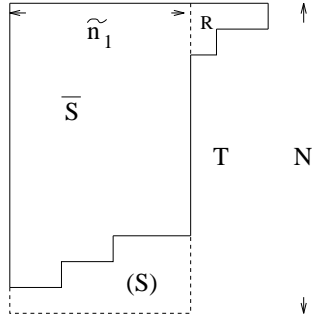


fig. 7.1: composite representation

We observe that all “interesting” IRREPs will look that way and that putting small representations R, S at “different ends” of T means that their variation produces independently new composite representations. Because we are looking for a perturbation theory we may sum over R, S independently as long as the leading contributions come soon, i.e. $N = 3$ is large enough.

What makes the composite representations really interesting is the fact that the weight in the sum almost (approximately) factorises, only giving one more, quite simple coupling term:

- Using the column-form of the Casimir-eigenvalue we get by simple substitution:

$$C_2(T) = C_2(R) + C_2(S) + \frac{2\tilde{n}n}{N} \tag{7.2}$$

- The dimension is not that easy to determine, but using the hook-formula (eg. [37]p.78) and observing that $\dim(\bar{S}) = \dim(S)$ by similarity it is easy to show that:

$$\dim(T) = \dim(R)\dim(S)Q(O(N^0)) \quad (7.3)$$

where $Q(O(N^0))$ is a correction term which has the limit 1 for $N \rightarrow \infty$. The zeroth order is all what will be used in the subsequent treatment¹.

Following this discussion we write the partition function as:

$$\begin{aligned} Z(G, \lambda A, N) &\approx \sum_n' \sum_{\tilde{n}}' \sum_{R \in Y_n}' \sum_{S \in Y_{\tilde{n}}}' \\ &\dim(S) \exp\left(-\frac{\lambda A}{2N} C_2(S)\right) \dim(R) \exp\left(-\frac{\lambda A}{2N} C_2(R)\right) \\ &\exp\left(-\frac{\lambda A n \tilde{n}}{N^2}\right) \end{aligned} \quad (7.4)$$

where the prime indicates that the approximation will only be good for summations over small representations. In case N is not large (i.e. “finite”) this summation is a massive overcounting (cf. [55])

7.2 Orientation reversing tubes

Each chiral sector is already interpreted. All that has to be done here is to provide an interpretation for the “coupling term”

¹ D.Gross and W.Taylor give the formula

$$\begin{aligned} Q &= \prod_{i,j} \frac{(N+1-i-j)(N+1-i-j+n_i+\tilde{n}_j)}{(N+1-i-j+n_i)(N+1-i-j+\tilde{n}_j)} \\ &\approx 1 - \frac{n\tilde{n}}{N^2} + \frac{n\tilde{C}(S) + \tilde{n}\tilde{C}(R)}{N^3} + O(N^4) \end{aligned}$$

where the product is ver the rows of R, S , but make only use of the zeroth order. Furthermore in [56] they announce a “more useful” formula for the dimension.

The term, of course, looks like the one we interpreted as a tube in the chiral sum. But taking a tube like in the chiral theory is not consistent with the idea of composite representations given by two independent sectors. This forbids the usage of perturbative tools from the sectors.

In the topology of surfaces the “connection” of two surfaces is usually done by removing a disc from both and connecting the resulting one-spheres with a cylinder. This procedure has the advantage that the result, the “connected sum”, is again a surface and has the same orientability character as the cartesian product ([22]p.58). But we do not want to do algebraic topology, we want to find a suitable, consistent formalism.

What the cylinder in the connected sum actually does is to map one one-sphere onto the other in a continuous way. This looks like a “homotopy”. As long as our “homotopy” is locally diffeomorphic everywhere we will not be able to give an interpretation different from the tubes used for the chiral sectors.

Taking the cylinder $C = S^1 \times [0, 1] = \{(z, x) : |z| = 1, 0 \leq x \leq 1\}$ and comparing the maps

$$\nu_-(z, x) = z(1 - 2x)$$

$$\nu_+(z, x) = z(1 - 2x) \quad \text{for } x \leq \frac{1}{2}$$

$$= \bar{z}(2x - 1) \quad \text{for } x \geq \frac{1}{2}$$

we can illustrate what a orientation preserving and a orientation reversing tube is:

- ν_+ :

z indicates some sense of circulation in the complex plane, \bar{z} the opposite one. In the middle ($x = \frac{1}{2}$) we don't have any sense of circulation; the orientation switches its sign.

- ν_- :

Here the orientation remains the same all the way.

Geometrically (see figure 7.2):

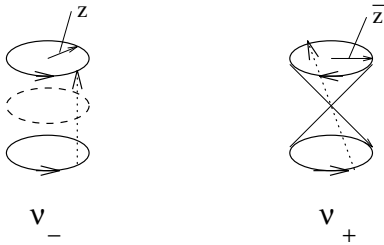


fig. 7.2: orientation reversing tubes

As before one imagines the genus to be increased by one and the combinatorial factor is, of course, $n\tilde{n}\lambda A$. Nice is that each such tube carries a minus sign. Because these orientation reversing tubes are local they exponentiate.

7.3 Free energy

Be Σ_G the set of disconnected covering maps then we have:

$$\begin{aligned}
Z(G, \lambda A, N) = & \\
& \sum_{\nu \in \Sigma_G} \frac{(-)^{\tilde{t}}}{|S_\nu|} \exp\left(-\frac{n\lambda A}{2}\right) \frac{(\lambda A)^{(i+t+\tilde{t}+h)}}{i! t! \tilde{t}! h!} \times \\
& N^{n(2-2G)-i-2(t+\tilde{t}+h)} \left[1 + O\left(\frac{1}{N}\right)\right]
\end{aligned} \tag{7.5}$$

where n is the winding number, i : “branching” number, t : number of orientation preserving tubes, \tilde{t} : number of orientation reversing tubes, h : number of infinitesimal handles, $\frac{1}{|S_\nu|}$: the symmetry factor, $O(\frac{1}{N})$: subleading contributions from the dimension (only for $G \neq 1$).

Some argumentation like in QFT may apply to show that the free energy is the corresponding sum of connected coverings ($\tilde{\Sigma}_G$) only:

$$\begin{aligned}
\ln Z = W(G, \lambda A, N) = & \\
& \sum_{\nu \in \tilde{\Sigma}_G} \frac{(-)^{\tilde{t}}}{|S_\nu|} \exp\left(-\frac{n\lambda A}{2}\right) \frac{(\lambda A)^{(i+t+\tilde{t}+h)}}{i! t! \tilde{t}! h!} \times \\
& N^{n(2-2G)-i-2(t+\tilde{t}+h)} \left[1 + O\left(\frac{1}{N}\right)\right]
\end{aligned} \tag{7.6}$$

Chapter 8

Frobenius' formulae, boundaries & gluing

The content of this chapter is already a little bit beyond the scope of this work, because it rather the subject of D.Gross' and W.Taylor's second article [56] than of their first one. Nevertheless I shall outline the development at the end of their first article.

The concepts used in this chapter have already been introduced and exemplified before:

- The “holonomies” associated with boundaries.
- The “gluing” of partition functions of areas with boundaries along edges using the orthogonality of the characters.
- The way of interpreting the $1/N$ expansion using branched coverings, calculating the symmetry factor, attaching tubes and handles.

For the discussion of coverings of bordered surfaces the introduction of a new set of class functions (the “power sums” [37], the “loop variables” [29]) is convenient. This will be the subject of the first section.

8.1 Frobenius' formulae

In the further development of the stringy approach to QCD_2 another set of class functions is used (see eg. [55] [56] [29] [57]):

Be $\sigma \in S_n$ with cycles of length $\{n_i\}_{i \in k}$, $\sum n_i = n$ then it is defined:

$$\Upsilon_\sigma(U) \equiv \prod_{j=1}^k (\text{tr} U^j)^{n_j} \quad (8.1)$$

These functions, in [29] referred to as “loop variables”, form a complete set of class functions (see “gluing” formula later on).

They are a form of the “power sums” defined in eg.[37]p.48; because $U \in U(N)$ is diagonalisable and the trace is invariant under similarity transformations they are, in effect, the same.

The “loop variables” Υ_σ satisfy relations with the characters χ_R , called the “Frobenius relations”:

$$\Upsilon_\sigma(U) = \sum_R \chi_R(\sigma) \chi_R(U) \quad (8.2)$$

This is a form of the Frobenius character formula [37]p.49¹.

There is an “inverse” relation, too:

$$\chi_R(U) = \sum_{\sigma \in S_n} \frac{1}{n!} \chi_R(\sigma) \Upsilon_\sigma(U) \quad (8.3)$$

which can be found by [37]p.460& 534.

Using these relations, some identities stated in the proof of the connection $\tilde{C}(R) \leftrightarrow \chi_R(T)$ we readily find the “gluing” formulae for loop variables as eg.

$$\begin{aligned} \int dU \Upsilon_\sigma(U) \Upsilon_\tau(U^+) &= \sum_R \chi_R(\sigma) \sum_{R'} \chi_{R'}(\tau) \int dU \chi_R(U) \chi_{R'}(U^+) \\ &= \sum_R \chi_R(\sigma) \chi_R(\tau) = \delta_{[\sigma], [\tau]} \frac{n!}{|[\sigma]|} \end{aligned} \quad (8.4)$$

¹ The connection between the “Schur polynomials” and the characters $\chi_R(U)$ is given in theorem 6.3, same reference

where $[\sigma]$ is the conjugacy class of σ and $||[\sigma]||$ is the number of elements in $[\sigma]$.

For the treatment of the nonchiral sum the loop variables are generalised for composite representations (see [29]5.5 [56]App.1). A thorough discussion of the interpretation of the nonchiral sum in terms of “coupled loop functions” is not possible in this work.

But in the the interpretation of the chiral sum the loop variables prove to be particularly well suited to the discussion of coverings with boundaries. This is basically because “during” the covering map the boundaries of the covering space wind around the boundaries of the target space several times, which is reflected by the different powers in the definition of the loop variables $\Upsilon_\sigma(U)$.

It is, therefore, necessary to express the partition function in terms of the loop variables and then to give a matching interpretation.

8.2 Interpretation: branched covering of a plaquette

The partition function of a plaquette with area A and holonomy U is:

$$Z_\Delta(U) = \sum_n \sum'_{R \in Y_n} \dim(R) \chi_R(U) \exp\left(-\frac{\lambda A}{2N} C_2(R)\right) \quad (8.5)$$

The following treatment is very analogous to the interpretation of the partition function for closed surfaces with genus G . Because the term $-\frac{n^2}{N}$ in the Casimir-eigenvalue does not give rise to any new argumentation it will be dropped for this discussion.

So the $1/N$ - expansion we have to interpret is:

$$Z_{\Delta}(U) = \sum_n \exp(-\frac{n\lambda A}{2}) \sum_i \frac{(\lambda A)^i}{i!} \times \sum_R [\frac{d_R}{n!} (-\frac{\tilde{C}(R)}{2})^i \chi_R(U) N^{n-i} + O(N^{n-i-1})] \quad (8.6)$$

Inserting the Frobenius character formula:

$$Z_{\Delta}(U) = \sum_n \exp(-\frac{n\lambda A}{2}) \sum_i \frac{(\lambda A)^i}{i!} \times \sum_{\sigma} [\frac{(-)^i}{|[\sigma]|} (\sum_R \frac{d_R|[\sigma]|}{n!^2} (\frac{\tilde{C}(R)}{2})^i \chi_R(\sigma) \Upsilon_{\sigma}(U) N^{n-i}) + O(N^{n-i-1})] \quad (8.7)$$

We define:

$$\phi_{ni}^{\sigma} \equiv \sum_R \frac{d_R|[\sigma]|}{n!^2} (\frac{\tilde{C}(R)}{2})^i \chi_R(\sigma) \quad (8.8)$$

As for areas with no boundaries this quantity will be interpreted as sum over branched covers with the number of distinct homomorphisms as symmetry factor.

The classification of surfaces with boundaries was described in chapter 6. With this the discussion for branched coverings of Δ is as before: We define $\Sigma_{\sigma}(n, i)$ to be the set of n -fold covers of Δ with “branching” number i and boundary of k disjoint one spheres S^1 which cover the boundary of Δ $\{n_i\}$ times. The $\{n_i\}$ are the sizes of the k cycles in the permutation σ .

Just as in chapter 6:

$$\begin{aligned} \sum_{\nu \in \Sigma_{\sigma}(n, i)} \frac{1}{|S_{\nu}|} &= \sum_{\tau_1 \dots \tau_i \in T} \frac{|[\sigma]|}{n!} \delta(\tau_1 \dots \tau_i \sigma) \\ &= \sum_R \frac{d_R|[\sigma]|}{n!^2} (\frac{\tilde{C}(R)}{2})^i \chi_R(\sigma) \end{aligned} \quad (8.9)$$

We can now rewrite the partition function as sum over coverings:

$$Z_{\Delta}(U) = \sum_{\nu} [\frac{(-)^i}{|S_{\nu}|} \exp(-\frac{n\lambda A}{2}) \frac{(\lambda A)^i}{i!} \prod_j (tr \hat{U}_j) N^{n-i} + O(N^{n-i-1})] \quad (8.10)$$

where the \hat{U}_j are the pullback-holonomies of the gauge field to the covering space². The \hat{U}_j are the reason for using the loop variables rather than the characters in the partition function.

8.3 Some outlooking remarks: On gluing in covering space

The discussion of the previous section may serve as an example how the covering of bordered surfaces works. By gluing plaquettes we get more complicated surfaces. In order to extent this discussion to the general case one has to determine the behaviour of coverings under gluing of the target space. In any case it is necessary to find a formalism for the gluing in order to make the interpretation complete because the invariance under subtriangulation/ the additivity of the partition function is one of the essential features of QCD_2 . The solution to this problem is the subject of the extensive work published in [56] and [57], and will therefore not be given here. Nevertheless I want to outline where about the next step leads.

Gluing involves integrals like

$$\begin{aligned}
& \int dU \frac{[[\sigma]]}{n!} \Upsilon_\sigma(VU) \frac{[[\tau]]}{n!} \Upsilon_\tau(U^+W) \\
&= \sum_{\sigma' \in [\sigma]} \sum_{\tau' \in [\tau]} \sum_R \frac{\chi_R(\sigma') \chi_R(\tau')}{\dim(R)} \chi_R(VW) \\
&= \sum_{\sigma' \in [\sigma]} \sum_{\tau' \in [\tau]} \sum_{\alpha \in S_n} N^{-n} \delta(\sigma' \tau' \alpha \Omega_n^{-1}) \frac{\Upsilon_\alpha(VW)}{n!}
\end{aligned} \tag{8.11}$$

where

$$\Omega_n = \sum_{\mu \in S_n} \left(\frac{1}{N}\right)^{n-K_\mu} \mu$$

²for pullback of holonomies see eg. [22]p.45& 201

and K_μ is the number of cycles in μ ([29](5.13)(5.34)).

Ω_n has to be introduced to express the partition function once again in terms of loop variables.

The gluing of edges of the cover is associated with the closely related integral [29](5.36)³

$$\int dU U_{j_1}^{i_1} .. U_{j_n}^{i_n} U_{l_1}^{+k_1} .. U_{l_n}^{+k_n} = \sum_{\rho\sigma} \frac{\delta(\Omega_n^{-1}\rho)}{N^n} \delta_{l_{\sigma(1)}}^{i_1} \delta_{j_1}^{k_{(\rho\sigma)(1)}} .. \delta_{l_{\sigma(n)}}^{i_n} \delta_{j_n}^{k_{(\rho\sigma)(n)}} \quad (8.12)$$

The δ_β^α express the gluing of an edge of the sheet α with one of the sheet β .

The Ω_n^{-1} are then objects of new interpretation procedures ([29][57][56]).

The nonchiral sum includes a new “interaction” between the two sectors due to the fact that $\chi_{\bar{S}R}(U) \neq \chi_S(U^+)\chi_R(U)$.

Because the conjugate representation \bar{S} is associated with traces over the adjoint of the holonomies U^+ the loop variables provide a further understanding how the two different orientations come into the theory.

D.Gross and W.Taylor [4] argue that the interaction term $\sim \frac{n\tilde{n}}{N}$ would cancel the contributions from folds occurring during the gluing edges of sheets with opposite orientation. These matters are furtherly discussed in the later publications.

³The formula for the proof of the character gluing formula in chapter 2 is this one for $n = 1$

8.4 Outview

It proved to be impossible to give a “complete” account of the progress made in this direction so far. Still I want to indicate where else results may be found.

The program was continued by D.Gross and W.Taylor in [56] and S.Cordes et al. in [57]. The latter seem to have been able to connect the interpretation with topological string theory. Furthermore they announce another approach to the string theory interpretation using harmonic covering maps rather than holomorphic ones by P. Horava [58].

Interesting papers on YM_2 are for example one by Blau & Thompson [59] and one by Witten [60]. In the latter YM_2 is discussed on nonorientable surfaces.

In the 1980’s much effort has been made to derive a resonable $1/N$ -expansion in equations for loops (“loop dynamics”) See for example [10].

The high energy behaviour of strings is discussed eg. by D.Gross and P.Mende [61]. They find a gaussian falloff at high energies which is incompatible with the observed powerlike behaviour of strong cross sections at high energies.

Nevertheless Verlinde & Verlinde made a recently updated effort to describe high energy QCD in terms of two dimensional nonabelian shock waves [62].

Finally I would like to mention the work by J.Baez and W.Taylor [55]. They give a description of QCD_2 for finite N by usage of loop variables and a secondly quantised Hamiltonian formalism.

Conclusions

Free QCD_2 on a compact surface is solved exactly. The $1/N$ -expansion of the related partition function for closed surfaces can be given as a “string” theory. This formalism can incorporate boundaries.

The question whether $1/N$ is a good perturbative parameter can not be answered before a generalisation to higher dimension and explicit calculations of eg. bound state energies.

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Bibliography

- [1] D.J.Gross: “Two dimensional QCD as a string theory”
Nucl.Phys.B400 (1993) 161
- [2] G.’t Hooft: “A planar diagram theory for strong interactions”
Nucl. Phys.B72 (1974) 461
- [3] J.Minahan: Phys.Rev.D47 (1993) 3430
“Summing over inequivalent maps in the string theory interpretation of two dimensional QCD”
- [4] D.J.Gross, W.Taylor: “Two dimensional QCD is a string theory”
Nucl.Phys.B400 (1993) 181
- [5] G.’t Hooft: “A twodimensional model for mesons”
Nucl.Phys.B75 (1974) 461
- [6] P.D.B.Collins, E.J.Squires: “Regge Poles in Particle Physics”
Ergebn. der exakt. Naturw. Vol.45, Springer Verlag, 1968
- [7] K.Wilson: “Confinement of Quarks” Phys.Rev.D10 (1974) 2445
- [8] H.Rothe: “Lattice Gauge Theories” World Scientific, Singapore, 1992
- [9] M.Creutz: “Quarks, Gluons and Lattices” CUP, 1983

- [10] A.Migdal: “Loop equations and $1/N$ -expansion”
Phys.Rep102 (1983) 199
- [11] M.Green, J.Schwarz& E.Witten “Superstring theory” Vol.I, CUP, 1987
- [12] E.D’Hoker, D.Phong: “The Geometry of String Perturbation Theory”
Rev.Mod.Phys.60 (1988) 917
- [13] A.Migdal: “Recursion equations in gauge field theories”
Sov.Phys.JETP 42 (1976) 413
- [14] B.Rusakov: Mod.Phys.Lett.A5 (1990) 693
“Loop averages and partition functions in $U(N)$ gauge theory on two dimensional manifolds”
- [15] L.Ryder: “Quantum Field Theory” CUP, 1985
- [16] D.Bailin, A.Love: “Introduction to Gauge Field Theory” IOP, 1993
- [17] J.Taylor: “Gauge Theories of Weak Interactions” CUP, 1976
- [18] P.Ramond: “Field theory: A modern primer” Benjamin-Cummings, 1981
- [19] S. Grossmann: “Funktionalanalysis” Aula, Wiesbaden, 1970
- [20] S.Okubo: J.Math.Phys.18 (1977) 2382
“Casimir invariants and vector operators in simple and classical Lie algebras”
- [21] J.Cornwell: “Group theory in physics” AP, 1984
- [22] M.Nakahara: “Geometry, Topology and Physics” A.Hilger, 1990
- [23] W.Fischer, I.Lieb: “Funktionentheorie”
Vieweg, Braunschweig, 1985

- [24] F.Constantinescu: Teubner, Stuttgart, 1974
 “Distributionen und ihre Anwendungen in der Physik”
- [25] S.Flügge: “Mathematische Methoden der Physik I”
 Springer Verlag, Berlin, 1979
- [26] M.Göckeler, T.Schücker:
 “Differential geometry, gauge theories, and gravity” CUP, 1987
- [27] C.Itzykson, J.Drouffe: “Statistical field theory” Vol.I CUP
- [28] P. Mansfield: Durham Preprint DTP-93-43 (to appear in Nucl.Phys.B)
- [29] S.Cordes, G.Moore, S. Ramgoolam:
 “Lectures on 2D Yang-Mills Theory, Equivariant Cohomology and Topological String theory”
 Lectures given at Springschool Trieste, 11.-19.04.1994, unpublished
- [30] Wu-Ki Tung: “Group theory in Physics” WS, 1985
- [31] L.Nachbin: “The Haar Integral” Van Nostrand, Princeton, 1965
- [32] R.Wilcox: J.Math.Phys.8 (1967) Nr4
 “Exponential Operators and Parameter Differentiation in Quantum Physics”
- [33] W.Miller: “Symmetry Groups and Their Applications” AP, 1972
- [34] C.Itzykson, J-B.Zuber: “Quantum Field Theory” McGraw-Hill, 1980
- [35] G.West: “Confinement, the Wilson loop and the Gluon Propagator”
 Phys.Lett.115B (1982) 468
- [36] J-P.Serre: “Linear Representations of Finite Groups” GTM42
 Springer Verlag, New York, 1977

- [37] W.Fulton, J.Harris: “Representation Theory” GTM129
Springer Verlag, New York, 1991
- [38] B.Wybourne: “Classical Groups for Physicists”
Wiley, New York, 1974
- [39] G.Fischer: “Lineare Algebra” Vieweg, Braunschweig, 1986
- [40] A.Messiah: “Quantenmechanik” Bd.2 de Gruyter, Berlin, 1985
- [41] M.Resnikoff: “A useful relation among Casimir operators”
Phys.Lett. 19 (1965) 596
- [42] R.Gilmore: J.Math.Phys. 11 (1970) 1855
“Spectrum of Casimir Invariants for the Simple Classical Lie Groups”
- [43] H.Weyl: “The Classical Groups” Princeton, 1946
- [44] G.Murtaza, M.Rashid: J.Math.Phys. 14 (1973) 1196
“Duality of a Young diagram describing a representation and dimensionality formulas”
- [45] G.Racah: “Group theory and spectroscopy” CERN 1961
- [46] Broecker, tom Dieck: “Representations of compact Lie Groups” GTM98
Springer Verlag, New York, 1985
- [47] C.Itzykson, M.Nauenberg: Rev.Mod.Phys. 38 (1966) 95
“Unitary Groups: Representations and Decompositions”
- [48] L.O’Raifeartaigh: “Group structure of gauge theories” CUP, 1986
- [49] J.Stillwell: Springer-Verlag, NY, 1980
“Classical Topology and Combinatorial Group Theory” GTM72

- [50] H.Farkas, I.Kra: “Riemann surfaces” GTM71,
Springer- Verlag, NY, 1992
- [51] L.Ahlfors, L.Sario: “Riemann surfaces”, Princeton, 1960
- [52] H.Zieschang, E.Vogt, H-D.Coldewey: Springer-Verlag, Berlin, 1980
“Surfaces and planar discontinuous Groups” lect.notes 835
- [53] W.Fulton: Ann.ofMath.90 (1969) 542
“Hurwitz schemes and irreducibility of moduli of algebraic curves”
- [54] C.Ezell: Trans.Am.Math.Soc. 243 (1978) 123
“Branch point structure of covering maps onto nonorientable surfaces”
- [55] J.Baez, W.Taylor: “Strings and Two-dimensional QCD for finite N”
preprint MIT-CTP-2266, hep-th/9401041
- [56] D.Gross, W.Taylor: Nucl.Phys.B403 (1993) 395
“Twists and Wilson loops in the string theory of two dimensional QCD”
- [57] S.Cordes, G.Moore, S.Ramgoolam: hep-th/9401107, YCTP-P23-93, RU-94-
20
“Large N 2D Yang-Mills Theory and Topological string theory”
- [58] P.Horava: hep-th/9311156, EFI-93-66
“Topological Strngs and QCD in Two Dimensions”
- [59] M.Blau, G.Thompson: “Quantum Yang-Mills on Arbitrary Surfaces”
Internat.J.of Mod.Phys.A7 (1992) 3781
- [60] E.Witten: “On Quantum Gauge Theories in Two Dimensions”
Comm.Math.Phys.141 (1991) 153

- [61] D.Gross, P.Mende: Phys.Lett.B197 (1987) 129
“The High Energy Behaviour of String Scattering Amplitudes”
- [62] H.Verlinde, E. Verlinde: PUPT-1319
“QCD at High Energies and Two-Dimensional Field Theory”
- [63] F.Wegner: J.Math Phys.12 (1971) 2259
- [64] Y.Aharonov, D.Bohm: Phys.Rev.115 (1959) 484
- [65] Xiao-Gang He, B.McKellar: Phys.Lett.B256 (1991) 250
- [66] H.Boersch, H.Hamisch, K.Grohmann u.D.Wohlleben:
Z.Phys.165 (1961) 79
- [67] N.Osakabe et.al.: Phys.Rev.A34 (1986) 815